A GEOMETRIC INVARIANT OF A FINITE GROUP

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ABSTRACT. We study the class of the classifying stack of a finite group in a Grothendieck group of algebraic stacks introduced previously. We show that this class is trivial in a number of examples most notably for all symmetric groups. We also give some examples where it is not trivial. The latter uses counterexamples of Saltman and Swan to the problem of Noether.

In [Ek09a] a Grothendieck group of algebraic stacks was introduced and was identified with a localisation of the Grothendieck group of algebraic varieties. The purpose of this article is to investigate the classes of stacks which can be considered to be at the opposite end of algebraic varieties, the classifying stacks of finite groups.

After having given some general results on the class in the Grothendieck group $K_0(\mathrm{Stck}_{\mathbf{k}})$ of algebraic stacks of the classifying stacks of finite groups we introduce the main idea that shall be used to obtain information about such classes. It is to use a faithful linear representation of the group that can be exploited in an inductive approach involving the subgroups that are the stabilisers of vectors of the representation. This is then applied to the symmetric groups to show that the classes of their classifying stacks are equal to 1. In the proof a lambda structure on $K_0(\mathrm{Stck}_{\mathbf{k}})$ using "stacky" symmetric powers plays a crucial rôle. In characteristic zero we then continue and show that that lambda structure also can be described as a certain natural extension of the lambda structure on the Grothendieck group of varieties defined using ordinary symmetric powers.

We go on to provide some non-triviality results. Our first result in that direction is that the unramified Brauer group of the invariant field of a faithful linear representation of the group (introduced in [Sa84] and identified cohomologically in [Bo97]) is an invariant of the class of the classifying stack and hence the examples of Saltman and Bogomolov give examples of groups for which the class is different from 1 (over any field of characteristic 0). Going backwards in time we then consider the case of the base field being the rationals and try to fit in an example of Swan. Using the existence of a smooth toric compactification of a (not necessarily split) torus we show that the class in $K_0(\mathrm{Stck}_{\mathbf{Q}})$ of the classifying stack of $\mathbf{Z}/47$ is non-trivial.

It is very clear from the arguments involved in both the triviality and non-triviality that they are closely related with invariant theory for finite groups. We finish by a short discussion on a possible actual connection.

1 Preliminaries

We recall some results from [Ek09a]. We defined a Grothendieck group $K_0(\mathrm{Stck}_{\mathbf{k}})$ of algebraic stacks of finite type over a field $\mathbf{k}$ with affine stabilisers. It is generated by isomorphism classes $\{X\}$ of such stacks with relations $\{X \setminus Y\} = \{X\} - \{Y\}$ for a closed substack $Y \subseteq X$ and $\{E\} = \{X \times \mathbf{A}^n\}$ for a vector bundle $E \to X$ of rank n. It becomes a ring under $\{X\}\{Y\} = \{X \times Y\}$ and the inclusion of the category of $\mathbf{k}$-schemes of finite type into that of algebraic $\mathbf{k}$-stacks then induces a ring homomorphism $K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Stck}_{\mathbf{k}})$, where $K_0(\mathrm{Spc}_{\mathbf{k}})$ is the usual Grothendieck group of $\mathbf{k}$-schemes. This induces (cf., [loc. cit., Thm. 1.3]) an isomorphism $K_0(\mathrm{Spc}_{\mathbf{k}})' \to K_0(\mathrm{Stck}_{\mathbf{k}})$, where $K_0(\mathrm{Spc}_{\mathbf{k}})'$ is obtained from $K_0(\mathrm{Spc}_{\mathbf{k}})$ by inverting $\mathbb{L} := \{\mathbf{A}^1\}$ and $\mathbb{L}^n - 1$ for all $n > 0$. This result is then used to get a map from $K_0(\mathrm{Stck}_{\mathbf{k}})$ to the completion $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ of $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ in the topology given by the dimension filtration. In particular, as $\mathbb{L}$ and $\mathbb{L}^n - 1 = \mathbb{L}^n(1 - \mathbb{L}^{-n})$ are invertible in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$, we get a factorisation $K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Stck}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ of the

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completion map. The map $K_0(\mathrm{Stck}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ takes $\mathrm{Fil}^n K_0(\mathrm{Stck}_{\mathbf{k}})$, the subgroup spanned by stacks of dimension $\leq n$, into $\mathrm{Fil}^n \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$, the closure of the subgroup spanned by the $\{X\}/\mathbb{L}^m$, for schemes X of dimension $\leq n+m$ (cf., [loc. cit., Prop. 2.1]).

As is well-known, one can define an Euler characteristic with compact support on $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ (see [loc. cit.] for a discussion on how this is done over a general field). The recipient ring $K_0(\mathrm{Coh}_{\mathbf{k}})$ for such an Euler characteristic with compact support is a Grothendieck group either of mixed Galois representations or of mixed Hodge structures. The important facts are that we get a ring homomorphism $\chi_c\colon K_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Coh}_{\mathbf{k}})$, that $K_0(\mathrm{Coh}_{\mathbf{k}})$ is a graded ring, graded by cohomological weight, and that χ_c extends to a ring homomorphism $\chi_c\colon \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$, where $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ is the completion of $K_0(\mathrm{Coh}_{\mathbf{k}})$ in the weight filtration, i.e., infinite sums in the direction of negative weights are admitted.

We let $\mathcal{Z}\mathrm{ar}$ be the class of (connected) algebraic group schemes of finite type all of whose torsors over any extension field of $\mathbf{k}$ are trivial. Another basic result is that if $H \subseteq G$ is a closed subgroup scheme of the finite type subgroup scheme $G \in \mathcal{Z}\mathrm{ar}$, then (cf., [loc. cit., Prop. 1.1 ix])

$$\{\mathrm{B}H\} = \{G/H\}\{\mathrm{B}G\} \tag{1}$$

and when $G \in \mathcal{Z}\mathrm{ar}$ we have (cf., [loc. cit., Prop. 1.3])

$$1 = \{G\}\{\mathrm{B}G\}. \tag{2}$$

Furthermore, for any finite dimensional $\mathbf{k}$-algebra L its group scheme of units belongs to $\mathcal{Z}\mathrm{ar}$ (cf., [loc. cit., Prop. 1.3]). To make later formulas explicit we shall also use the formula (cf., [loc. cit., Prop. 1.1])

$$\{\mathrm{GL}_n\} = (\mathbb{L}^n - \mathbb{L}^{n-1}) \cdots (\mathbb{L}^n - 1). \tag{3}$$

If G acts on an algebraic space, the passage to an invariant subspace commutes with taking quotients under G *provided* the order of G is invertible in the base field $\mathbf{k}$. That is not true in general but the following proposition shows that in a (very) special case it is so.

Proposition 1.1 *i) Let X be an algebraic $\mathbf{k}$-space of finite type and Y a closed subspace with complement U. Then for every $0 \leq m \leq n$, the image of $Y^m \times U^{n-m} \subseteq X^n$ under the quotient map $X^n \to X^n/\Sigma_n$ is isomorphic to $Y^m/\Sigma_m \times U^{n-m}/\Sigma_{n-m}$.*

ii) Let R be an equivalence relation on $\{1, 2, \dots, n\}$ and let $\mathbf{A}^R$ be the subscheme of $\mathbf{A}^n$ defined by $i \sim_R j \implies x_i = x_j$. If Σ_R is the subgroup of Σ_n consisting of the permutations preserving R (i.e., $i \sim_R j \implies \sigma i \sim_R \sigma j$) then the image of $\mathbf{A}^R$ under the quotient map $\mathbf{A}^n \to \mathbf{A}^n/\Sigma_R$ is isomorphic to the quotient $\mathbf{A}^R/\Sigma_R$.

Proof: For i) it is clear that we get an induced map $Y^m/\Sigma_m \times U^{n-m}/\Sigma_{n-m} \to X^n/\Sigma_n$ and it will be enough to show that it is an immersion. Now, it is clear that it is an injection on geometric points so it is enough to show that it is unramified. Consider the induced map $f_{n,m}\colon X^n/\Sigma_m \times \Sigma_{n-m} \to X^n/\Sigma_n$. As the Σ_n-stabiliser of any point of $Y^m \times U^{n-m} \subseteq X^m$ lies in $\Sigma_m \times \Sigma_{n-m}$ we get that $f_{n,m}$ is étale along $Y^m \times U^{n-m}$ and hence it is enough to show that $Y^m/\Sigma_m \times U^{n-m}/\Sigma_{n-m} \to X^n/\Sigma_m \times \Sigma_{n-m}$ is unramified. This morphism is the product of an identity map and the map $Y^m/\Sigma_m \to X^n/\Sigma_m$. The latter map is however a closed immersion. This is well-known and the reason is the following: The problem is local so we may assume that X is affine. Then the result follows from the fact that if $U \to V$ is a surjective map of $\mathbf{k}$-vector spaces, then the induced map $(U^{\otimes n})^{\Sigma_n} \to (V^{\otimes n})^{\Sigma_n}$ is surjective and in fact a splitting of $U \to V$ induces an equivariant splitting of $U^{\otimes n} \to V^{\otimes n}$.

As for ii) let Σ^R be the normal subgroup of Σ_R fixing R (i.e., the elements that fix the R-equivalence classes). We start by identifying $\mathbf{A}^n/\Sigma^R$ and the image of $\mathbf{A}^R$ in it. In the case of characteristic zero everything is clear as then taking group quotients commutes with passing to an (invariant) subscheme so we may assume that the base field $\mathbf{k}$ has characteristic $p > 0$. If r is an R-equivalence class we put $s_r := |r|$ and write $s_r = p^{k_r} s'_r$ where $p \not| s'_r$. We then have that $\Sigma^R = \prod_r \Sigma_{s_r}$ (the product running over all R-equivalence classes) acting diagonally on

$\mathbf{A}^n = \prod_r \mathbf{A}^{s_r}$. Writing $\mathbf{A}^{s_r} = \mathbf{Spec}\,\mathbf{k}[x_{r,i}]_{1\le i\le s_r}$ we get that $\mathbf{A}^n/\Sigma^R = \mathbf{Spec}\,\mathbf{k}[\sigma_{r,i}]$, where the $\sigma_{r,i}$ are the elementary symmetric functions of the $x_{r,i}$. Furthermore, the composite $\mathbf{Spec}\,\mathbf{k}[x_r] = \mathbf{A}^R \to \mathbf{A}^n \to \mathbf{A}^n/\Sigma^R$ is induced by the graded ring homomorphism given by $\sum_i \sigma_{r,i} \mapsto (1+x_r)^{s_r}$. This implies that the image of $\mathbf{A}^R$ in $\mathbf{A}^n/\Sigma^R$ is equal to $\mathbf{Spec}\,\mathbf{k}[x_r^{p^{k_r}}]$. Now, Σ_R acts on $\mathbf{A}^n/\Sigma^R$ through the quotient Σ_R/Σ^R. This latter group is realised as the group of permutations of the R-equivalence classes that preserve the cardinalities of the classes. Hence, as $\mathbf{A}^n/\Sigma^R \cong \prod_r \mathbf{A}^{s_r}$ with Σ_R/Σ^R permuting the factors we get that $\mathbf{A}^n/\Sigma_R$ is the product of symmetric powers of the $\mathbf{A}^{s_r}$. Furthermore, the image of $\mathbf{A}^R$ in $\mathbf{A}^n/\Sigma_R$ is, as we just saw, a product of closed embeddings of $\mathbf{A}^1$'s in the $\mathbf{A}^{s_r}$. Hence we may conclude by the fact that symmetric powers commute with closed embeddings which was proved in i). $\square$

2 The stacky and non-stacky lambda structures

Recall that we have a lambda ring string structure on $K_0(\mathrm{Spc}_\mathbf{k})$ for which $\sigma_t(x)$ (the operation defined by $\sigma_t(x)\lambda_{-t}(x) = 1$) is given by $\sigma_t(\{X\}) = \sum_{n\ge 0}\{\sigma^n(X)\}t^n$, where $\sigma^n(X) := X^n/\Sigma_n$. (However, the references I know of assume that the characteristic is zero. Using (1.1:i) that restriction can be lifted.) We further put, for algebraic spaces X and Y, $\sigma^n_Y(X) := X^n \times_{\Sigma_n} \mathrm{Conf}^n(Y)$, where $\mathrm{Conf}^n(Y) := \{(y_i) \in Y \mid \forall i \ne j\colon y_i \ne y_j\}$. The third part of the following proposition is [Gö01, Lemma 4.4] and the rest of it is proved in essentially the same way. We shall however need the full result and hence give the complete proof.

Proposition 2.1 *Let X and Y be algebraic $\mathbf{k}$-spaces of finite type and assume that $\mathbf{k}$ has characteristic 0 or $> n$.*

i) We have that

$$\sigma^n(\{X\}\{Y\}) = \sum_{\lambda\vdash n}\prod_i\{\sigma_Y^{n_i}(\sigma^{\lambda_i}(X))\},$$

where $\lambda = [\lambda_1^{n_1}, \lambda_2^{n_2}, \dots, \lambda_k^{n_k}]$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ runs over the partitions of n.

ii) There is a universal polynomial (depending only on n) in variables $x_{m,\mu}$ where $\mu = (\mu_1, \dots, \mu_r)$ runs over sequences of positive integers which gives $\sigma^n_Y(X)$ when evaluated at $x_{m,\mu} = \sigma^m(\sigma^\mu(\{X\})\{Y\})$ where $\sigma^\mu(\{X\}) := \sigma^{\mu^1}(\sigma^{\mu^2}(\dots(\{X\})))$. In particular σ^n_Y can be naturally extended to $K_0(\mathrm{Spc}_\mathbf{k})$.

iii) We have that $\sigma^n(\mathbb{L}x) = \mathbb{L}^n\sigma^n(x)$ for all $x \in K_0(\mathrm{Spc}_\mathbf{k})$.

PROOF: Given an equivalence relation R on a finite set S and an algebraic space Y we define the subspace Y^S_R of Y^S by

$$Y^S_R := \{(y_s) \in Y^S \mid y_s = y_t \iff s \sim_R t\}.$$

(In particular $\mathrm{Conf}^n(Y) = Y^n_\Delta$, where $\Delta \subseteq \{1,\dots,n\}^2$ is the diagonal.) Then Y^n is the disjoint union of the Y^n_R where R runs through all the equivalence relations on $\{1,\dots,n\}$ and hence $(X\times Y)^n$ is the disjoint union of the $X^n \times Y^n_R$. The orbits under Σ_n of the R correspond to partitions of n and if let Y^n_λ be the union of the Y^n_R over the orbit corresponding to $\lambda = [\lambda_1^{n_1}, \lambda_2^{n_2}, \dots, \lambda_k^{n_k}]$, then $(X^n\times Y^n_\lambda)/\Sigma_n$ is isomorphic to $(X^n\times Y^n_R)/N_R$, where R is the equivalence relation with equivalence classes $\{1,\dots,\lambda_1\}$, $\{\lambda_1+1,\dots,2\lambda_1\}$,…,$\{(n_1-1)\lambda_1+1,\dots,n_1\lambda_1\}$ $\{n_1\lambda_1+1,\dots,n_1\lambda_1+\lambda_2\}$ etc and N_R is the group of permutation fixing R. We have that N_R is the product

$$N_R = (\Sigma_{\lambda_1} \int \Sigma_{n_1}) \times \cdots \times (\Sigma_{\lambda_k} \int \Sigma_{n_k})$$

which implies that $(X^n \times Y^n_R)/N_R$ is isomorphic to

$$(\sigma^{\lambda_1}(X))^{n_1} \times_{\Sigma_{n_1}} \mathrm{Conf}^{n_1}(Y) \times \cdots \times (\sigma^{\lambda_k}(X))^{n_k} \times_{\Sigma_{n_k}} \mathrm{Conf}^{n_k}(Y)$$

which gives i). Then ii) follows from i) by induction.

Finally, applying i) to $X = \mathbf{Spec}\,\mathbf{k}$ gives

$$\sigma^n(\{Y\}) = \sum_{\lambda \vdash n} \prod_i \{\sigma_Y^{n_i}(1)\}$$

and applied to $X = \mathbf{A}^1$, and using that $\sigma^m(\mathbf{A}^1) = \mathbf{A}^m$, it gives

$$\sigma^n(\mathbb{L}\{Y\}) = \sum_{\lambda \vdash n} \prod_i \{\sigma_Y^{n_i}(\mathbf{A}^{\lambda_i})\}.$$

Now, Σ_m acts freely on $\mathrm{Conf}^m(Y)$ and acts linearly on $(\mathbf{A}^k)^m$ so that $(\mathbf{A}^k)^m \times_{\Sigma_m} \mathrm{Conf}^m(Y)$ is a vector bundle over $\mathrm{Conf}^m(Y)/\Sigma_m$ and hence $\{\sigma_Y^m(\mathbf{A}^k)\} = \mathbb{L}^{mk}\{\sigma_Y^m(1)\}$. This gives

$$\sum_{\lambda \vdash n} \prod_i \{\sigma_Y^{n_i}(\mathbf{A}^{\lambda_i})\} = \sum_{\lambda \vdash n} \prod_i \mathbb{L}^{\lambda_i n_i} \sigma_Y^{n_i}(1) = \mathbb{L}^n \sum_{\lambda \vdash n} \prod_i \{\sigma_Y^{n_i}(1)\} = \mathbb{L}^n \sigma^n(\{Y\})$$

which proves iii) in the case when $x = \{Y\}$. Now, iii) can be expressed as the equality $\sigma_t(\mathbb{L}x) = \sigma_{\mathbb{L}t}(x)$ and the multiplicativity of both sides shows that if it is true for $x = \{Y\}$, then it is always true. □

Remark: i) The formula $\sigma^n(\mathbb{L}x) = \mathbb{L}^n\sigma^n(x)$ for all n or the equivalent $\lambda^n(\mathbb{L}x) = \mathbb{L}^n\lambda^n(x)$ for all n would follow from the fact that $\lambda_t(\mathbb{L}) = 1 + \mathbb{L}t$ if $K_0(\mathrm{Spc}_{\mathbf{k}})$ were a *special* lambda ring (which it isn't).

ii) The condition $p > n$ is probably necessary. We have at least that the inverse image of a stratum of $\sigma^n(X)$ under the map $\sigma^n(X \times \mathbf{A}^1) \to \sigma^n(X)$ is not necessarily a vector bundle. This is seen already for $p = n = 2$ and the "diagonal" stratum.

We are going to use this proposition to extend this lambda structure to both $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ and $K_0(\mathrm{Stck}_{\mathbf{k}})$. For the latter we need the following lemma.

Lemma 2.2 *Let R be lambda ring and $r \in R$ an element such that $\lambda_t(rx) = \lambda_{rt}(x)$ for all $x \in R$ and $\lambda_t(1) = 1+t$. If S is a set of integer polynomials closed under multiplication and substitution $t \mapsto t^n$ for all $n > 0$ and $S' := \{\, f(r) \mid f \in S \,\}$, then there is a unique lambda ring structure on the localisation $R[S'^{-1}]$ for which $R \to R' := R[S'^{-1}]$ is a lambda ring homomorphism and for which $\lambda_t(rx) = \lambda_{rt}(x)$ for all $x \in R'$.*

Proof: Recall (cf., [SGA6, Exp. V, 2.3.2]) that one can define a ring structure on $1 + tA[[t]]$, functorial in the commutative ring A, with addition being multiplication of power series and the multiplication being characterised (apart from being functorial) by being continuous in the t-adic topology and fulfilling $\psi(t) \circ (1 + at) = \psi(at)$ for all elements $\psi(t)$. Hence the condition $\lambda_t(rx) = \lambda_{rt}(x)$ can be rephrased as saying that $\lambda_t(rx) = \lambda_t(r) \circ \lambda_t(x)$. This in particular shows that the restriction of λ_t to the subring U generated by r is a ring homomorphism and the requirement of $\lambda_t(rx) = \lambda_{rt}(x)$ for an extension to R' would make λ_t a ring homomorphism on $U' := U[S'^{-1}]$ and the condition can then be rephrased as saying that λ_t should be a module homomorphism over U'. Hence, the uniqueness is clear and for existence it would be enough to show that the image in $1 + tR'[[t]]$ of $\lambda_t(f(r)) \in 1 + tR[[t]]$ is invertible for all $f \in S$. Write $f(x) = \sum_i n_i x^i$ and let $\psi(t) = 1 + \sum_{n>0} a_n t^n$ be a putative inverse so that we should have $\lambda_t(f(r)) \circ \psi(t) = 1 + t$. As λ_t is a ring homomorphism on U this expands to

$$\prod_i \psi(r^i t)^{n_i} = 1 + t$$

and we want to show by induction over n that we may choose the $a_n \in R'$ so that this equality is valid. Now, the t^n-coefficient of the left hand side equals $f(r^n)a_n +$(pol. in the $a_i, i < n,$ and r) so that the condition of equality for the t^n coefficient may be written

$$f(r^n)a_n = \text{pol. in the } a_i, i < n, \text{ and } r$$

and by assumption $f(r^n)$ has been inverted in R'. □

This lemma and the result of Totaro mentioned above now gives an extension of the lambda ring structure on $K_0(\mathrm{Spc}_{\mathbf{k}})$.

Theorem 2.3 *The lambda ring structure on $K_0(\mathrm{Spc}_{\mathbf{k}})$ has an extension to a lambda ring structure on $K_0(\mathrm{Spc}_{\mathbf{k}})'$ and $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ characterised by $\lambda_t(\mathbb{L}x) = \lambda_{\mathbb{L}t}(x)$ for all $x \in K_0(\mathrm{Spc}_{\mathbf{k}})'$ (resp. $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$). On $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ it is continuous in the dimension filtration and thus extends to a lambda ring structure on $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$.*

PROOF: The extensions to $K_0(\mathrm{Spc}_{\mathbf{k}})'$ and $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ follow directly from Proposition 2.1 and Lemma 2.2. One then shows that if $\dim x \leq n$ for $x \in K_0(\mathrm{Spc}_{\mathbf{k}})$, then $\dim \lambda^i(x) \leq in$ by the fact that this is obvious when $x = \{X\}$ with $\dim X \leq n$ and then it follows for general elements. This and the definition of the extension of the λ^i to $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ then shows that $\dim \lambda^i(x) \leq i \dim x$ remains true for $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ which implies continuity. □

Using the identification $K_0(\mathrm{Spc}_{\mathbf{k}})' = K_0(\mathrm{Stck}_{\mathbf{k}})$ we in particular get a lambda ring structure on $K_0(\mathrm{Stck}_{\mathbf{k}})$ though somewhat indirectly defined. There is however a lambda ring structure which looks more natural. It uses the symmetric stack powers instead of ordinary symmetric powers (the latter do not of course make sense for algebraic stacks). Now, stack quotients of groups acting on stacks can be somewhat tricky (particularly if the action is non-strict). We thus start by taking by taking some time to describe the symmetric stack powers in concrete terms.

- We recall the definition of the *wreath product*, $\mathcal{E} \int \mathrm{B}\Sigma_n$ (which is a special case of wreath products of categories) of a category $\mathcal{E}$ and the category $\mathrm{B}\Sigma_n$. Its objects are n-tuples $(e_1, \ldots, e_n)$ of objects of $\mathcal{E}$. Morphisms from $(e_1, \ldots, e_n)$ to $(e'_1, \ldots, e'_n)$ are tuples $(\sigma, f_1, \ldots, f_n)$ where $\sigma \in \Sigma_n$ and $f_i \colon e_i \to e'_{\sigma^{-1}(i)}$. Composition is given by

$$(\tau, g_1, \ldots, g_n) \circ (\sigma, f_1, \ldots, f_n) = (\tau\sigma, g_{\sigma^{-1}(1)} \circ f_1, \ldots, g_{\sigma^{-1}(n)} \circ f_n).$$

  If we have $\mathcal{E} = \mathrm{B}G$ for a group G then we may also form the wreath product of groups $G \int \Sigma_n$ and we have $\mathrm{B}(G \int \Sigma_n) = \mathrm{B}G \int \mathrm{B}\Sigma_n$. More generally if the group G acts on X and $[X/G]$ is the action groupoid $G \times X \underset{m}{\overset{p_2}{\rightrightarrows}} X$, then $[X/G] \int \mathrm{B}\Sigma_n = [X^n/(G \int \Sigma_n)]$ (with the obvious action of $G \int \Sigma_n$ on X^n).

- $\mathcal{E} \int \mathrm{B}\Sigma_n$ clearly commutes with fibre products but not in general with 2-fibre products.[1] Note however that there are cases when a fibre product is a 2-fibre product. Notably, we say that a functor $F \colon \mathcal{A} \to \mathcal{C}$ is a *fibration* if it is surjective on isomorphisms. It is then easy to show that if $\mathcal{A} \to \mathcal{C}$ is a fibration and $\mathcal{B} \to \mathcal{C}$ is any functor, then the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a 2-fibre product. Now, it is equally easy to see that if $\mathcal{A} \to \mathcal{C}$ is a fibration, then the induced functor $\mathcal{A} \int \mathrm{B}\Sigma_n \to \mathcal{C} \int \mathrm{B}\Sigma_n$ is also a fibration.

- If $\mathcal{G}$ is a pseudo-functor, then we can define a new pseudo-functor $\mathcal{G} \int \mathrm{B}\Sigma_n$ given by $(\mathcal{G} \int \mathrm{B}\Sigma_n)(U) = \mathcal{G}(U) \int \mathrm{B}\Sigma_n$. If $\mathcal{G}$ is a stack then we can define the n'th *symmetric stack power*, $\mathcal{S}\mathrm{ymm}^n\mathcal{G}$ by taking the stack associated to this pseudo-functor. It is easy to see that if $\mathcal{G}$ is a pseudofunctor with $\mathcal{G}'$ as its associated stack then the stack associated to $\mathcal{G} \int \mathrm{B}\Sigma_n$ is naturally equivalent to $\mathcal{S}\mathrm{ymm}^n\mathcal{G}'$. (Note that $\mathcal{S}\mathrm{ymm}$ preserves 2-fibre squares where one of the maps is just a local fibration, i.e., locally surjective on morphisms.)

- If $\mathcal{E}$ is an algebraic stack and $\mathcal{G}$ is a chart for it, i.e., a groupoid scheme with smooth source and target maps whose associated stack is equivalent to $\mathcal{E}$ then $\mathcal{G} \int \mathrm{B}\Sigma_n$ is a chart for $\mathcal{S}\mathrm{ymm}^n\mathcal{E}$.

We start by a fairly simple result that however will be very important to us.

Lemma 2.4 *Assume that $\mathcal{X}$ is an algebraic stack and $\mathcal{E} \to \mathcal{X}$ a vector bundle of rank d. Then $\mathcal{S}\mathrm{ymm}^n\mathcal{E} \to \mathcal{S}\mathrm{ymm}^n\mathcal{X}$ is a vector bundle of rank nd.*

[1] To us that means that 2-fibre squares should be commutative up to an equivalence. We shall however only use it for groupoids where this is the same as commuting up to a natural transformation.

PROOF: We may choose a chart $\mathcal{G} = (\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0)$ of $\mathcal{X}$ such that $\mathcal{E}$ is trivial over $\mathcal{G}_0$. This means that we have a functor $\mathcal{G} \to \mathrm{BGL}_d$ and if we let $\mathcal{G}'$ be the fibre product

$$\begin{array}{ccc} \mathcal{G}' & \longrightarrow & [\mathbf{A}^n/\mathrm{GL}_d] \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathrm{BGL}_d \end{array}$$

then it is a chart for $\mathcal{E}$ as $[\mathbf{A}^d/\mathrm{GL}_d] \to \mathrm{BGL}_d$ is a fibration so that this is also a 2-fibre square. Applying the wreath product we get a cartesian square

$$\begin{array}{ccc} \mathcal{G}' \wr \mathrm{B}\Sigma_n & \longrightarrow & [\mathbf{A}^d/\mathrm{GL}_d] = [(\mathbf{A}^d)^n/(\mathrm{GL}_d \wr \Sigma_n)] \\ \downarrow & & \downarrow \\ \mathcal{G} \wr \mathrm{B}\Sigma_n & \longrightarrow & \mathrm{BGL}_d \wr \mathrm{B}\Sigma_n = \mathrm{B}(\mathrm{GL}_d \wr \Sigma_n). \end{array}$$

However, the action of $\mathrm{GL}_d \wr \Sigma_n$ on $(\mathbf{A}^d)^n = \mathbf{A}^{dn}$ is linear so that $[(\mathbf{A}^d)^n/(\mathrm{GL}_d \wr \Sigma_n)] \to \mathrm{B}(\mathrm{GL}_d \wr \Sigma_n)$ is a vector bundle (induced by the group homomorphism $\mathrm{GL}_d \wr \Sigma_n \to \mathrm{GL}_{dn}$) and therefore so is $\mathcal{S}\mathrm{ymm}^n(\mathcal{E}) \to \mathcal{S}\mathrm{ymm}^n(\mathcal{X})$. $\square$

We are now prepared for defining the more natural lambda structure on $K_0(\mathrm{Stck}_{\mathbf{k}})$.

Proposition 2.5 *There is a lambda structure $\{\lambda^n_s\}$ on $K_0(\mathrm{Stck}_{\mathbf{k}})$ with the property that $\sigma^n_s(\{\mathcal{C}\}) = \{\mathcal{S}\mathrm{ymm}^n\mathcal{X}\}$ for any algebraic stack $\mathcal{X}$. We have that $\lambda^n_s(\mathbb{L}x) = \mathbb{L}^n\lambda^n_s(x)$.*

PROOF: We have to verify that the defining relations for $K_0(\mathrm{Stck}_{\mathbf{k}})$ are verified for the map $\mathcal{X} \mapsto \sigma^s_t(\mathcal{X}) := \sum_n \{\mathcal{S}\mathrm{ymm}^n\mathcal{X}\}t^n$. That it is constant on isomorphism classes is clear. If $\mathcal{E} \to \mathcal{X}$ is a vector bundle of rank d, then by Lemma2.4 we have that $\mathcal{S}\mathrm{ymm}^n\mathcal{E} \to \mathcal{S}\mathrm{ymm}^n\mathcal{X}$ is a vector bundle of rank dn so that $\{\mathcal{S}\mathrm{ymm}^n\mathcal{E}\} = \{\mathcal{S}\mathrm{ymm}^n\mathcal{X} \times \mathbf{A}^{dn}\}$. If we apply this also to the trivial vector bundle $\mathcal{X} \times \mathbf{A}^d \to \mathcal{X}$ this shows that $\sigma^s_t(\mathcal{E}) = \sigma^s_t(\mathcal{X} \times \mathbf{A}^d)$.

Consider now the situation where $\mathcal{Y} \subseteq \mathcal{X}$ is a closed substack of $\mathcal{X}$. We have to verify that $\sigma^s_t(\mathcal{X}) = \sigma^s_t(\mathcal{X} \setminus \mathcal{Y})\sigma^s_t(\mathcal{Y})$. This is done in the same way as for the lambda structure on $K_0(\mathrm{Spc}_{\mathbf{k}})$ (cf., [LL04, Lemma 3.1]).

The last part follows from the previously established formula, $\{\mathcal{S}\mathrm{ymm}(\mathcal{X}\times\mathbf{A}^1)\} = \{\mathcal{S}\mathrm{ymm}(\mathcal{X})\times \mathbf{A}^n\}$. $\square$

3 Classifying stacks of finite group schemes

We start by getting some formulas for the class $\mathrm{B}G$ where G is a finite group scheme.

Proposition 3.1 *Let V be an n-dimensional $\mathbf{k}$-vector space and $G \subseteq \mathrm{GL}(V)$ a finite subgroup scheme.*

i) $\{\mathrm{B}G\} = \{\mathrm{GL}(V)/G\}/\{\mathrm{GL}(V)\}$.

ii) The image of $\{\mathrm{B}G\}$ in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ is equal to $\lim_{m\to\infty}\{V^m/G\}/\mathbb{L}^{mn}$.

iii) $\chi_c(\{\mathrm{B}G\}) = 1$.

PROOF: The first part is a special case of (1). For the second part we define U_m to be the open subset of V^m consisting of the sequences $v_1, \dots, v_m$ for which one of the sequences $v_1, \dots, v_n$, $v_{n+1}, \dots, v_{2n}$, $\dots$, $v_{kn-n+1}, \dots, v_{kn}$, where $k := \lfloor m/n \rfloor$, form a basis for V. Then U_m is $\mathrm{GL}(V)$-invariant and $\mathrm{GL}(V)$ acts freely on it. Furthermore, the codimension of $V_m := V^m \setminus U_m$ in V^m is equal to k and hence tends to ∞ with m and we have

$$\begin{aligned} \{\mathrm{B}G\} - \{V^m/G\}/\mathbb{L}^{mn} = \{[V^m/G]\}/\mathbb{L}^{mn} - \{V^m/G\}/\mathbb{L}^{mn} = \\ \{[U_m/G]\}/\mathbb{L}^{mn} + \{[V'_m]\}/\mathbb{L}^{mn} - \{U_m/G\}/\mathbb{L}^{mn} - \{[V_m/G]\}/\mathbb{L}^{mn} = \\ \{[V'_m/G]\}/\mathbb{L}^{mn} - \{V_m/G\}/\mathbb{L}^{mn} \overset{m\to\infty}{\longrightarrow} 0, \end{aligned}$$

where V'_m is the complement of U_m/G in V^m/G and we have used that as G acts freely on U_m we have $[U_m/G] \cong U_m/G$.

From the first part we get $\chi_c(\{\mathrm{B}G\}) = \chi_c(\{\mathrm{GL}(V)/G\})/\chi_c(\{\mathrm{GL}(V)\})$ and by definition

$$\chi_c(\{\mathrm{GL}(V)/G\}) = \sum_i (-1)^i \{H^i_c(\mathrm{GL}(V)/G)\}$$

and the same for $\mathrm{GL}(V)$. Now, as the coefficient for cohomology is a field of characteristic zero we have that $H^i_c(\mathrm{GL}(V)/G) = H^i(\mathrm{GL}(V))^{G(\bar{\mathbf{k}})}$. However, the right multiplication action of G on $\mathrm{GL}(V)$ extends to $\mathrm{GL}(V)$ and $\mathrm{GL}(V)$ is a connected algebraic group so all elements of $\mathrm{GL}(V)(\bar{\mathbf{k}})$ acts trivially on $H^i_c(\mathrm{GL}(V))$ and hence we get $H^i_c(\mathrm{GL}(V)/G) = H^i_c(\mathrm{GL}(V))$ and in particular $\chi_c(\{\mathrm{GL}(V)/G\}) = \chi_c(\{\mathrm{GL}(V)\})$. □

Remark: i) With small modifications of the proof and with V^m/G interpreted as the GIT-quotient $\mathbf{Spec}\,\mathbf{k}[V^m]^G$ the first two results remain true for arbitrary G.

ii) In general $\chi_c(x)$ contains a lot of information on x so the fact that $\chi_c(\{\mathrm{B}G\}) = 1$ should make the equality $\{\mathrm{B}G\} = 1$ our first guess.

iii) In [Ek09a] a completion $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ of $K_0(\mathrm{Spc}_{\mathbf{k}})$ in a stronger topology than the dimension filtration was defined and a homomorphism $K_0(\mathrm{Stck}_{\mathbf{k}}) \to \overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$ was shown to exist. However, the natural map $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}}) \to \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ turned out to be injective so we don't lose any information by computing the limit $\{V^m/G\}/\mathbb{L}^{mn}$ in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ rather than in $\overline{K}_0^{\mathrm{pol}}(\mathrm{Spc}_{\mathbf{k}})$. Though I haven't checked the details it seems that this sequence actually converges in the stronger topology.

The first formula is generally difficult to use as the variety $\mathrm{GL}(V)/G$ can be quite complicated. There are however some cases when it (or rather(1)) can be used.

Proposition 3.2 *i) We have that* $\{\mathrm{B}\mu_n\} = 1$ *in* $K_0(\mathrm{Stck}_{\mathbf{k}})$ *for any field* $\mathbf{k}$.

ii) If $\mathbf{k}$ *is a finite field and* $G \in \mathcal{Z}\mathrm{ar}_k$*, then* $\{\mathrm{B}G(\mathbf{k})\} = 1$.

PROOF: We have a natural embedding $\mu_n \subseteq \mathbf{G}_m$ and applying (1) using that $\mathbf{G}_m \in \mathcal{Z}\mathrm{ar}$ we get that $\{\mathrm{B}\mu_n\} = \{\mathbf{G}_m/\mu_n\}\{\mathrm{B}\mathbf{G}_m\}$ in $K_0(\mathrm{Stck}_{\mathbf{k}})$. Now, we have that $\mathbf{G}_m/\mu_n = \mathbf{G}_m$ and by (2) we have $\{\mathbf{G}_m\}\{\mathrm{B}\mathbf{G}_m\} = 1 \in K_0(\mathrm{Stck}_{\mathbf{k}})$.

As for the second part, the Lang torsor $G \to G$ given by $g \mapsto Fg \cdot g^{-1}$ gives an isomorphism $G/G(\mathbf{k}) \cong G$ and we conclude by (1). □

Example: The case of $\mathbf{Z}/n$ is more complicated. Writing $\mathbf{Z}/n$ as a product of its primary components reduces to the case of n being a power of a prime p (as $\mathrm{B}(G \times H) = \mathrm{B}(G) \times \mathrm{B}(H)$). If p divides the characteristic we can embed $\mathbf{Z}/p^k$ into the truncated Witt vector scheme $\mathbf{W}_m$ as the kernel of $F - 1$. Hence, $\mathbf{W}_m/(\mathbf{Z}/p^k) \cong \mathbf{W}_m$ and as $\mathbf{W}_m \in \mathcal{Z}\mathrm{ar}$ we get $\{\mathbf{Z}/p^k\} = 1$. (This is a special case of Proposition 3.2.) We are hence left with the case when p is invertible in $\mathbf{k}$. Generally when n is invertible in $\mathbf{k}$ we can consider the separable $\mathbf{k}$-algebra $L := \mathbf{k}[x]/(\Phi_n(x))$, where $\Phi_n(x)$ is the n'th cyclotomic polynomial. We let T be the torus of invertible elements in L. The residue ζ of x is a unit of order n giving an injective map $\mathbf{Z}/n \hookrightarrow T$. As $T \in \mathcal{Z}\mathrm{ar}$ by (1) and (2) we get that $\{\mathrm{B}\mathbf{Z}/n\} = \{T/(\mathbf{Z}/n)\}\{T\}^{-1}$. Now, the character group of T, as $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-module, is $\mathbf{Z}[\mu_n^*]$, where μ_n^* is the set of primitive n'th roots of unity in an algebraic closure $\bar{\mathbf{k}}$ of $\mathbf{k}$. The inclusion $\mathbf{Z}/n \hookrightarrow T$ is dual to the map $\mathbf{Z}[\mu_n^*] \to \mu_n$ which is the inclusion $\mu_n^* \hookrightarrow \mu_n$ on generators. Hence, the character group of $T/(\mathbf{Z}/n)$ is the kernel I_n of this map, a Galois module first considered by Swan (cf., [Sw69]).

i) Let us first consider only the case when n is a prime p. Swan notices that $I = I_p$ is a locally free module over the group ring $\mathbf{Z}[\mathrm{Aut}(\mu_p)] \cong \mathbf{Z}[\mathbf{Z}/(p-1)]$ (this is true as the localisation equals $\mathbf{Z}[\mu_p \setminus \{1\}]$ at primes different from p and the order of the Galois group is invertible at p). If I is actually free, then $T/(\mathbf{Z}/p)$ is isomorphic to T and we get $\{\mathrm{B}\mathbf{Z}/p\} = 1$. This is certainly is true if all rank one locally free $\mathbf{Z}[\mathbf{Z}/(p-1)]$-modules are free which in turn is true if the class group of $\mathbf{Z}[\mathbf{Z}/(p-1)]$ is trivial. (The converse is not true *a priori* but Swan

shows that for $p = 47$ I is not free.) Let us recall the standard analysis of the class group of the integer group ring of a cyclic group C_n of order n: We have an injection $\mathbf{Z}[C_n] \hookrightarrow \prod_{d|n} \mathbf{Z}[\zeta_d]$, where $\mathbf{Z}[\zeta_d]$ is the ring of integers in the field of d'th roots of unity. This inclusion induces a surjection on class groups so that in particular the class group $\mathbf{Z}[C_n]$ is trivial only if the class groups of the $\mathbf{Z}[\zeta_d]$ are trivial. The class group of $\mathbf{Z}[\zeta_d]$ is trivial when d is one of $1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 33, 35, 36, 40, 44, 45, 48, 60, 84$ or twice an odd number among them (as $\mathbf{Z}[\zeta_d] = \mathbf{Z}[\zeta_{2d}]$ if d is odd). Hence for an n not among these, the class number of $\mathbf{Z}[C_n]$ is greater than 1. The kernel of the map on class groups is then analysed in terms of units. We are going to show that $\mathbf{Z}[C_{p-1}]$ has class number 1 for $p = 2, 3, 5, 7, 11$ and class number greater than 1 for all other primes. For $p = 2$ this is trivial and for $p = 3$ it follows from the fact that the class number of $\mathbf{Z}[C_q]$ is equal to that of $\mathbf{Z}[\zeta_q]$ for a prime q, a result due to Rim (cf., [Ri59]). Similarly the inclusion $\mathbf{Z}[C_4] \hookrightarrow \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}[i]$ induces an isomorphism on class groups by [KM77, p. 416] which takes care of $p = 5$. As for $p = 7, 11$ we use Milnor's Mayer-Vietoris sequence (cf., [Mi71, §3, Thm. 3.3]): If n is odd we have $C_{2n} = C_2 \times C_n$ and we get an embedding $\mathbf{Z}[C_{2n}] \hookrightarrow \mathbf{Z}[C_n] \times \mathbf{Z}[C_n]$, where in the first factor we send the generator of C_2 to 1 and in the second to -1. The subring $\mathbf{Z}[C_{2n}]$ of the product is then characterised by then condition that a pair (a, b) belongs to it when the reductions of a and b modulo 2 coincide. From the Milnor sequence we get that the kernel of the map $\tilde{K}_0(\mathbf{Z}[C_{2n}]) \to \tilde{K}_0(\mathbf{Z}[C_n])^2$ is isomorphic to the cokernel of the map $(\mathbf{Z}[C_n])^* \to (\mathbf{Z}/2[C_n])^*$. If we can show that this cokernel is trivial for $n = 3, 5$ we conclude that $\mathbf{Z}[C_{2n}]$ has class number 1 again by Rim's theorem. For $n = 3$ we get that $\mathbf{Z}/2[C_3] = \mathbf{Z}/2 \times \mathbf{F}_4$ and already $C_3 \subset (\mathbf{Z}[C_3])^*$ maps surjectively onto $(\mathbf{Z}/2 \times \mathbf{F}_4)^* = \mathbf{F}_4^*$. We have similarly for $n = 5$ that $\mathbf{Z}/2[C_5] = \mathbf{Z}/2 \times \mathbf{F}_{16}$. Now, $(\mathbf{Z}/2 \times \mathbf{F}_{16})^* = C_{15} = C_3 \times C_5$ and $C_5 \subset (\mathbf{Z}[C_5])^*$ fills out the C_5-part. Furthermore, $x^3 + x^2 - 1$ is a unit in $\mathbf{Z}[C_5] = \mathbf{Z}[x]/(x^5 - 1)$ (with inverse $x^4 + x - 1$) and it maps to $(\zeta + 1)^2$ in $\mathbf{F}_{16} = \mathbf{Z}/2[x]/(x^4 + x^3 + x^2 + x + 1)$, where ζ is the residue of x. As $(\zeta + 1)^5 = (\zeta^4 + 1)(\zeta + 1) = \zeta^3 + \zeta^2 + 1 \neq 1$ we see that the image of $x^3 + x^2 - 1$ in $(\mathbf{Z}/2 \times \mathbf{F}_{16})^*$ has order divisible by 3 and hence the map $(\mathbf{Z}[C_5])^* \to (\mathbf{Z}/2[C_5])^*$ is surjective. As for showing that for all other p, the class group of $\mathbf{Z}[C_{p-1}]$ is non-trivial we need only look at primes for which the class number of $\mathbf{Z}[\zeta_{p-1}]$ is equal to 1 which gives us a finite list. This list can be treated with a suitable computer algebra system.[2] (We are only going to use the case of triviality of $\{\mathrm{B}\mathbf{Z}/p\}$.)

ii) Consider now the case $n = 4$. Then $I = I_4$ has a basis consisting of $[i] - [-i]$ and $2[i] + 2[-i]$ and is hence isomorphic as a Galois module to $\mathbf{Z} \times \mathbf{Z}i$. Now, $\mathbf{Z}i$ is the character group of $T/\mathbf{G}_m$ so that $T/(\mathbf{Z}/4)$ is isomorphic to $\mathbf{G}_m \times T/\mathbf{G}_m$. As $T \to T/\mathbf{G}_m$ is a $\mathbf{G}_m$-torsor we get that $\{T/(\mathbf{Z}/4)\} = \{\mathbf{G}_m\}\{T/\mathbf{G}_m\} = \{\mathbf{G}_m\}\{T\}\{\mathbf{G}_m\} = \{T\}$ and hence $\{\mathrm{B}\mathbf{Z}/4\} = 1$.

Example: We assume (for simplicity) that $\mathbf{k}$ is algebraically closed.

i) Let G be a finite subgroup of $\mathbf{G}_m$. Then we have just seen that $\{\mathrm{B}G\} = 1$ in $K_0(\mathrm{Stck}_{\mathbf{k}})$.

ii) Let G be a finite subgroup of the group of affine transformations (of $\mathbf{A}^1$). We shall show by induction on the order of G that $\{\mathrm{B}G\} = 1$ (in $K_0(\mathrm{Stck}_{\mathbf{k}})$ as always) and we assume that $|G| > 1$. If G fixes a point of $\mathbf{A}^1$ then it is conjugate to a subgroup of $\mathbf{G}_m$, a case that we have already treated so we assume that all stabilisers of points are proper subgroup. The stack quotient $[\mathbf{A}^1/G]$ is the $\mathbf{A}^1$-fibration associated to the universal G-torsor over $\mathrm{B}G$ so that by (1) we get that $\{[\mathbf{A}^1/G]\} = \mathbb{L}\{\mathrm{B}G\}$. Now, $\mathbf{A}^1$ is the disjoint union of a finite number of G-orbits $O_1, \ldots, O_n$ with non-trivial stabiliser and an open subset U where G acts freely. This gives that $\{[\mathbf{A}^1/G]\} = \sum_i \{[O_i/G]\} + [U/G]$. As G acts freely on U we have $[U/G] = U/G$ and we also have $[O_i/G] \cong \mathrm{B}G_i$ where G_i is the stabiliser of some point of O_i. By assumption each G_i is a proper subgroup of G and hence $\{\mathrm{B}G_i\} = 1$. This gives $\{[\mathbf{A}^1/G]\} = n + \{U/G\}$. Now, we have that $\mathbf{A}^1/G \cong \mathbf{A}^1$ and hence U/G is open in $\mathbf{A}^1$ with complement in bijection with the set of orbits $\{O_i\}$. Hence $\{U/G\} = \mathbb{L} - n$ and we get that $\{[\mathbf{A}^1/G]\} = \mathbb{L}$. As $\mathbb{L}$ is invertible in $K_0(\mathrm{Stck}_{\mathbf{k}})$ we conclude from $\mathbb{L} = \mathbb{L}\{\mathrm{B}G\}$ that $\{\mathrm{B}G\} = 1$.

iii) Let G be a finite subgroup of PGL_2. We shall show by induction on the order of G that

[2] See, `http://www.math.su.se/~teke/CyclicClassgroup.mg` for a Magma (cf., [BCP]) script that performs this computation.

$\{\mathrm{B}G\} = 1 \in K_0^{\mathrm{PGL}_2}(\mathrm{Stck}_{\mathbf{k}})$. If G has a fixed point in its action on $\mathbf{P}^1$, then it is conjugate to a subgroup of the group of affine transformations and hence we have already shown that $\{\mathrm{B}\} = 1$ so we may assume that all stabilisers are proper. As before we get that $\{[\mathbf{P}^1/G]\} = (\mathbb{L}+1)\{\mathrm{B}G\}$ (but now only in $K_0^{\mathrm{PGL}_2}(\mathrm{Stck}_{\mathbf{k}})$ as $\mathrm{PGL}_2 \notin \mathcal{Z}\mathrm{ar}$). We now proceed exactly as in the affine case (using that $\mathbf{P}^1/G \cong \mathbf{P}^1$) and conclude that $\{[\mathbf{P}^1/G]\} = \{\mathbf{P}^1\} = \mathbb{L}+1$. Now, $\mathbb{L}+1$, being a factor of $\mathbb{L}^2 - 1$ is invertible in $K_0(\mathrm{Stck}_{\mathbf{k}})$ so we conclude.

iv) Let G be the $g = 1$ theta group of level a prime p. This means that G is a non-trivial central extension of $\mathbf{Z}/pz$ by $\mathbf{Z}/p \times \mathbf{Z}/p$ (for $p = 2$ we get two possible groups, the dihedral and quaternion groups, depending on the parity of the theta characteristic). Assume that $\mathbf{k}$ contains p'th root of unity (and in particular $p \not| \operatorname{char} \mathbf{k}$). Then G has an (irreducible) representation V of dimension p where the central element z acts by a fixed primitive root of unity. We have that $\{[V/G]\} = \mathbb{L}^p\{\mathrm{B}G\}$. On the other hand, every non-central element of G has a line as fixed point locus. Each such line is stabilised by a subgroup $\mathbf{Z}/p \times \mathbf{Z}/pz$ and there are in all $p(p+1)$ such lines forming in all $p+1$ conjugacy classes under the action of G. If U is the complement of all those lines we have that G acts freely on U and hence $[U/G] \cong U/G$. The complement of $U \cup \{0\}$ is the union of the fixed point lines with the origin removed and it divides up into conjugacy classes under G. If V_L is such a conjugacy class containing the line L minus the origin, then $[V_L/G] \cong [(L \setminus \{0\})/G_L]$ where G_L is the stabiliser of L and is hence equal to $\mathbf{Z}/p \times \mathbf{Z}/pz$, where the first factor acts trivially and the second freely on $L \setminus \{0\}$. Hence

$$[(L \setminus \{0\})/G_L] \cong \mathrm{B}\mathbf{Z}/p \times (L \setminus \{0\})/(\mathbf{Z}/pz) \cong \mathrm{B}\mathbf{Z}/p \times \mathbf{G}_m$$

and as $\{\mathrm{B}\mathbf{Z}/p\} = 1$ we get

$$\mathbb{L}^p\{\mathrm{B}G\} = \{U/G\} + (p+1)(\mathbb{L}-1) + \{\mathrm{B}G\}.$$

If we also use that V/G is the disjoint union of U/G, $p+1$ copies of $\mathbf{G}_m$ and (the image of) $\{0\}$ we get

$$(\mathbb{L}^p - 1)\{\mathrm{B}G\} = \{V/G\} - 1.$$

As $\mathbb{L}^p - 1$ is invertible in $K_0(\mathrm{Stck}_{\mathbf{k}})$ this gives a formula for $\{\mathrm{B}G\}$ showing in particular that $\{\mathrm{B}G\} = 1$ precisely when $\{V/G\} = \mathbb{L}^p$ in $K_0(\mathrm{Spc}_{\mathbf{k}})'$.

The argument used in these examples can clearly be generalised. We shall only treat the case of the action of GL_n on $\mathbf{A}^n$. Hence let G be a finite group acting linearly on a finite dimensional $\mathbf{k}$-vector space V. For every subgroup H we let V_H be the variety of points of V whose stabiliser is exactly H. It is clear that the V_H for varying H form a stratification of V in the strong sense of being a disjoint decomposition and the closure of each stratum being the union of strata. More precisely, the closure of each non-empty V_H is equal to the fixed point subspace V^H and $V^H = \dot\cup_{T \subseteq H} V_T$. We say that H is a *stabiliser subgroup* (wrt to V) if V_H is non-empty. If $\mathcal{H}$ is a G-conjugacy class of subgroups of G, then we shall also denote by $V_{\mathcal{H}}$ the union of the V_H for $H \in \mathcal{H}$. We then have the following lemma.

Lemma 3.3 *i) When V_H is non-empty $N_G(H)/H$ acts freely on V_H and thus faithfully on V^H.*

ii) The stack quotient $[V/G]$ is the disjoint union of the locally closed substacks $[V_{\mathcal{H}}/G]$, where $\mathcal{H}$ runs over the conjugacy classes of subgroups of G.

iii) Each $[V_{\mathcal{H}}/G]$ is isomorphic to $[V_H/N_G(H)]$ for any $H \in \mathcal{H}$.

iv) If V_H is non-empty then V_H equals $(V^H)_H$ considered as an $N_G(H)$-representation.

PROOF: These are all immediate. □

This lemma opens up for an inductive approach to the computation of $\{\mathrm{B}G\}$: We have that $\{[V/G]\} = \mathbb{L}^{\dim V}\{\mathrm{B}G\}$ and we can write $[V/G]$ as the disjoint union of $[V_H/N_G(H)]$, where each $[V_H/N_G(H)]$ is also the top stratum of the stratification of $[V^H/N_G(H)]$. Note that this does not give a complete recursion as we are still left with the stratum corresponding to $H = \{e\}$. If however V is a faithful G-module, then the action of G on $V_e = V_{\{e\}}$ is free and hence $[V_e/G] = V_e/G$ giving at least a recursive formula for $\{\mathrm{B}G\}$ involving only $\{\mathrm{B}N\}$ for proper

subgroups of G and V_e/G. To prepare for the combinatorics that is involved in such a formula we need to introduce some notation. Hence we define the notion of *stabiliser flag* (with respect to the G-representation V and of length n) as follows: It is a sequence $f = (H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n)$ of G such that

- $H_0 = \{e\}$ and
- H_{i+1} is a stabiliser subgroup of the action of $\cap_{j\le i} N_G(H_j)$ on V^{H_i}.

We shall also use the notation $N_G(f)$ for the normaliser of the flag (i.e., $\cap_{j\le n} N_G(H_j)$), H_f for H_n, d_f for $\dim V^{H_f}$ and n_f for its length. Furthermore a flag is said to be *strict* if all inclusions $H_i \subseteq H_{i+1}$ are strict.

Theorem 3.4 *Let G be a finite group and V a faithful G-representation of dimension d. Then we have that*

$$\{\mathrm{B}G\}\mathbb{L}^d = \{V_e/G\} + \sum_f (-1)^{n_f}\{\mathrm{B}N_G(f)\}\mathbb{L}^{d_f},$$

where the sum runs over G-conjugacy class representatives of strict stabiliser flags of length ≥ 1.

PROOF: Note first that for an arbitrary finite group K and a finite dimensional K-representation U over $\mathbf{k}$ there is a disjoint decomposition of $[V/G]$ whose pieces are the $[V_H/N_K(H)]$, where H runs over the conjugacy classes of stabiliser subgroups. This gives, together with the fact that $[V/K] \to \mathrm{B}K$ is a vector bundle of rank d, the formula

$$\{\mathrm{B}K\}\mathbb{L}^d = \sum_H \{[V_H/N_K(H)]\},$$

where H runs over conjugacy class representatives of stabiliser subgroups. For $K = G$ and $U = V$ we can separate out $H = \{e\}$ and use that G acts freely on V_e as V is faithful to replace $[V_e/G]$ with V_e/G. This gives

$$\{\mathrm{B}K\}\mathbb{L}^d = \{V_e/K\} + \sum_H \{[V_H/N_K(H)]\},$$

where the sum now is over non-trivial H. On the other hand we may apply the formula with $K = N_G(f)$ and $U = V^{H_f}$ for a strict flag f and writing it as

$$\{[V_{H_f}/N_G(f)]\} = \{\mathrm{B}N_G(f)\}\mathbb{L}^{d_f} - \sum_{f'} \{[V_{H_{f'}}/N_G(f')]\},$$

where f' runs over strict stabiliser flags that extend f of length one more than that of f. Using this formula inductively to replace the summands in the sum gives the theorem. □

Remark: i) It is not true that the theorem always gives recursive formula for $\{\mathrm{B}G\}$ in $K_0(\mathrm{Spc}_{\mathbf{k}})'$. The reason is that $\{\mathrm{B}G\}$ will appear on the right hand side, always[3] for the flag $\{e\} \subset G$ but in general for any strict flag all of whose members are normal subgroups of G. Moving all of those summands to the right we get a formula for $\varphi(\mathbb{L})\{\mathrm{B}G\}$ in terms of $\{V_e/G\}$ and $\{\mathrm{B}H\}$ for proper subgroups, where φ is a monic integer polynomial of degree d. The problem is that φ may not be invertible in $K_0(\mathrm{Spc}_{\mathbf{k}})'$ (though there are many examples where it is). One may of course go further and invert all monic integer polynomials in $\mathbb{L}$.

ii) It is not difficult to extend the result to finite étale group schemes. (The only problem is that stabiliser subgroups may not be defined over the base field. This forces one to work with finite Deligne-Mumford stacks instead of just classifying stacks of finite groups.) For a general finite group scheme things are different however. The proof of the theorem uses that there are only a finite of stabiliser subgroups of G. For a connected finite group scheme this is not necessarily true.

The combinatorics of this formula easily become somewhat complicated. The following lemma allows us to bypass some of these complications.

[3] Except when $G = \{e\}$.

Lemma 3.5 *Two elements in $K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$ that are rational functions in $\mathbb{L}$ are equal when their images under χ_c are equal. In particular, for a finite group G we have that $\{\mathrm{B}G\} = 1$ in $K_0^{\mathcal{G}}(\mathrm{Stck}_{\mathbf{k}})$ if it is a rational function (with integer coefficients) in $\mathbb{L}$.*

Proof: The first part follows from the fact that the ring $\mathbf{Z}((q^{-1}))$ of Laurent power series in $q^{-1} = \chi_c(\mathbb{L}^{-1})$ embeds in $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ and that χ_c maps rational functions in $\mathbb{L}$ injectively into this image. The second part follows from the first as the rational cohomology of G is trivial we have that $\chi_c(\{\mathrm{B}G\}) = 1$ in $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$. □

To apply this result to the computation of $\{\mathrm{B}G\}$ we have to fight with the problem mentioned above. Thus for a G-representation V we define its *characteristic polynomial* $\varphi_V(x)$ as

$$x^d - \sum_f (-1)^{n_f} x^{d_f},$$

where $d = \dim V$ and f runs over all stabiliser flags normalised by G.

Proposition 3.6 *Let G be a finite group and V a faithful $\mathbf{k}$-representation and let $\varphi \in \mathbf{Z}[x]$ be a polynomial divisible by φ_V. Assume that $\phi(\mathbb{L})\{\mathrm{B}N_G(f)\} = \phi(\mathbb{L})$ in $K_0(\mathrm{Stck}_{\mathbf{k}})$ for all* proper *normalisers of stabiliser flags f. Then $\varphi(\mathbb{L})\{\mathrm{B}G\} = \varphi(\mathbb{L})$ precisely when $\phi(\mathbb{L})\{V_e/G\} = \phi(\mathbb{L})\mathbb{L}^{\dim V_e}$, where $\phi = \varphi/\varphi_V$.*

Proof: This follows immediately from Theorem 3.4 and Lemma 3.5. □

Note that $N_G(f)$ in general does not act freely on V_{H_f}, indeed H_f by definition acts trivially on it. However, $N_G(f)/H_f$ *does* act freely on V_{H_f}. As H_f also acts trivially on V^{H_f} we can apply our recursion also to this action. This gives the following result.

Proposition 3.7 *Suppose that the order of G is invertible in $\mathbf{k}$ and let $\phi \in \mathbf{Z}[x]$. Assume that $\phi(\mathbb{L})\varphi_{V^{H_f}}(\mathbb{L})\{\mathrm{B}N_G(f)\} = \phi(\mathbb{L})\varphi_{V^{H_f}}(\mathbb{L})$ for all* proper *normalisers of stabiliser flags f and $\phi(\mathbb{L})\{\mathrm{B}N_G(f)/H_f\} = \varphi(\mathbb{L})$ for all strict stabiliser flags f of length ≥ 1. Then $\phi(\mathbb{L})\varphi_V(\mathbb{L})\{\mathrm{B}G\} = \phi(\mathbb{L})\varphi_V(\mathbb{L})$ precisely when $\phi(\mathbb{L})\{V/G\}$ is a polynomial in $\mathbb{L}$. Furthermore, this is true precisely when $\phi(\mathbb{L})\{V/G\} = \phi(\mathbb{L})\mathbb{L}^d$, where $d = \dim V$.*

Proof: As the order of G is invertible we have, for U a G-invariant subscheme of V, that the quotient map $U/G \to V/G$ maps U/G isomorphically onto its image. This means that we get a stratification of V/G by the $V_H/N_G(H)$, where H runs over conjugacy class representatives of stabiliser subgroups of G. We of course also have that $V_H/N_G(H) = V_H/W_H$, where we have put $W_H := N_G(H)/H$ so that we get $\{V/G\} = \sum_H \{V_H/W_H\}$. By assumption and Proposition 3.6 we have that $\phi(\mathbb{L})\{V_H/W_H\}$ is a polynomial in $\mathbb{L}$ for $H \neq \{e\}$ and hence we get that $\phi(\mathbb{L})\{V/G\}$ is a polynomial in $\mathbb{L}$ precisely when $\phi(\mathbb{L})\{V_e/G\}$ is. Again by Proposition 3.6 we conclude that if $\phi(\mathbb{L})\{V/G\}$ is a polynomial in $\mathbb{L}$ then $\varphi(\mathbb{L})\{\mathrm{B}G\} = \varphi(\mathbb{L})$. Finally, as the cohomology of V/G are the G-invariants of the cohomology of V we get V and V/G have the same cohomology which implies that $\chi_c(\{V/G\}) = \chi_c(\mathbb{L}^d)$ and the last part follows from Lemma 3.5. □

3.1 Unipotent group schemes

We shall now show that if G is a unipotent group scheme over a field $\mathbf{k}$, then $\{\mathrm{B}G\} = \mathbb{L}^{-\dim G}$. In particular, if G is finite then $\{\mathrm{B}G\} = 1$, which fits in with the theme of the current section but the proof of the general case is not more difficult. Recall that a group scheme is *unipotent* if it can be embedded as a closed subgroup scheme of the group of strictly upper triangular matrices $U_n \subseteq \mathrm{GL}_n$ or equivalently every non-zero linear representation of it has a non-zero trivial subrepresentation. We say that a smooth unipotent group scheme is *split* if it can be obtained as a successive central extension of $\mathbf{G}_a$'s. In particular U_n is split (and over a perfect field all unipotent group schemes are split).

Proposition 3.8 *Let G be a split unipotent smooth group scheme over* $\mathbf{k}$ *and H a closed subgroup scheme. Then G/H is isomorphic to* $\mathbf{A}^{\dim G/H}$.

PROOF: We prove this by induction over $\dim G$, the case of dimension 0 being trivial. By assumption G contains a central subgroup $Z \subseteq G$ isomorphic to $\mathbf{G}_a$ and for which G/Z is split. Putting $Z' := Z \cap H$ we have a natural morphism $G/H \to (G/Z)/(H/Z')$ of homogeneous spaces. We have an action of Z on G/H given by left multiplication and as Z is central we get an induced action of Z/Z' making $G/H \to (G/Z)/(H/Z')$ a Z/Z'-torsor. Now, a quotient of $\mathbf{G}_a$ is either 0 or isomorphic to $\mathbf{G}_a$ again. Furthermore, the induction assumptions imply that $(G/Z)/(H/Z')$ is isomorphic to some $\mathbf{A}^n$ and all $\mathbf{G}_a$-torsors over $\mathbf{A}^n$ are trivial so that $G/H \cong \mathbf{A}^{n+1}$. □

This immediately gives the desired result.

Corollary 3.9 *Let G be a unipotent group scheme over the field* $\mathbf{k}$. *Then* $\{\mathrm{B}G\} = \mathbb{L}^{-\dim G}$.

PROOF: By assumption we can embed G in some split smooth unipotent group H. As $\mathcal{Z}$ar is closed under extensions we get that $H \in \mathcal{Z}$ar and therefore $\{\mathrm{B}G\} = \{H/G\}\{\mathrm{B}H\}$ and $1 = \{H\}\{\mathrm{B}H\}$. Applying the proposition to $e \subseteq H$ we get $H \cong \mathbf{A}^{\dim H}$ and by applying it to $G \subseteq H$ we get $H/G \cong \mathbf{A}^{\dim H/G}$ and combining we get $\{\mathrm{B}G\} = \mathbb{L}^{\dim(H/G)-\dim G} = \mathbb{L}^{-\dim G}$. □

Remark: i) A finite group scheme G is unipotent precisely when the augmentation ideal of the dual Hopf algebra $\mathbf{k}[G]^*$ is nilpotent. This applies in particular to when G is an étale group scheme whose order is a power of the characteristic of $\mathbf{k}$ and even more specifically when G is a constant p-group.

ii) If $\mathbf{k}$ is non-perfect there are smooth connected unipotent 1-dimensional group schemes G not isomorphic to $\mathbf{G}_a$ such as the kernel of the morphism $\mathbf{G}_a \times \mathbf{G}_a \to \mathbf{G}_a$ given by $(x, y) \mapsto x^p - x + ty^p$, where $t \in \mathbf{k}$ is not a p'th power. We then have that G is not isomorphic as $\mathbf{k}$-variety to $\mathbf{A}^1$ yet $\{\mathrm{B}G\} = \mathbb{L}^{-1}$. Of course it could still happen that $\{G\} = \mathbb{L}$ and hence $\{G\}\{\mathrm{B}G\} = 1$.

4 The symmetric groups

We now want to apply the theorem to show that $\{\mathrm{B}\Sigma_n\} = 1$ for all n. For that we shall use the standard permutation representation V_n of Σ_n. A quick way of using it to get a weaker statement is to use Proposition 3.1 by considering $\{(V_n)^m/\Sigma_n\}$. We note that $(V_n)^m$ is isomorphic as Σ_n-representation to $(\mathbf{k}^m)^n$ where the elements of Σ_n permutes the coordinates. Hence $(V_n)^m/\Sigma_n \cong \sigma^n(\mathbf{A}^m)$ and hence by Proposition 2.1 we have $\{(V_n)^m/\Sigma_n\} = \sigma^n(\mathbb{L}^m) = \mathbb{L}^{mn}$ and hence the image of $\{\mathrm{B}\Sigma_n\}$ in $\widehat{K}_0(\mathrm{Coh}_{\mathbf{k}})$ is equal to $\lim_{n\to\infty} \mathbb{L}^{mn}/\mathbb{L}^{mn} = 1$. Note that this argument works only when $n! \neq 0$ in $\mathbf{k}$ and as we don't know if $K_0(\mathrm{Stck}_{\mathbf{k}})$ is separated in the dimension topology we don't know how much information the equality gives about $\{\mathrm{B}\Sigma_n\}$ in $K_0(\mathrm{Stck}_{\mathbf{k}})$.

The idea for the full result is to use $V_n/\Sigma_n \cong \mathbf{A}^n$ combined with Theorem 3.4. The characteristic polynomial is $x^n - x$ which gives an invertible element of $K_0(\mathrm{Stck}_{\mathbf{k}})$. We would then like to use Proposition 3.7 assuming inductively triviality for the class of the classifying stack of stabiliser subgroups. This doesn't quite work for small characteristics but there (1.1:ii) comes to our rescue.

The inductive nature of the proof forces us to deal with groups more general than the symmetric groups. To prepare for this we want to interpret stabiliser flags in more combinatorial terms. Thus if S is a (finite) set, an *equivalence flag* on S is a sequence $R = (R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n)$ of equivalence relations on S with R_0 being equal to the diagonal in $S \times S$. It is *strict* if the inclusions are strict. The *flag stabiliser*, N_R, of the flag consists of all permutations of S which preserve the equivalence relations R_i. We let the *residue group* of R be the image of N_R in Σ_{S/R_n} so that it is isomorphic to N_R/H_R, where H_R is the subgroup of N_R consisting of the permutations that map each equivalence class of R_n into itself.

To the equivalence flag R we associate a sequence $H_0 \subseteq \cdots \subseteq H_n$ of subgroups of Σ_S, the group of permutations of S, namely H_i is the set of permutations mapping each equivalence class of R_i into itself. We then have the following lemma.

Lemma 4.1 *The above association of a sequence of subgroups of Σ_n to an equivalence flag gives an Σ_n-equivariant bijection between the set of equivalence flags on $\{1,2,\dots,n\}$ and the set of stabiliser flags of Σ_n with respect to its standard permutation representation. In particular the normaliser of a stabiliser flag is the group of permutations fixing all the R_i.*

PROOF: If we think of an element of the standard permutation representation as a function $\{1,2,\dots,n\} \to \mathbf{k}$ then a permutation is in the stabiliser of such an element precisely when it takes the equivalence classes of the equivalence relation on $\{1,2,\dots,n\}$ associated to such a function into themselves. From this the lemma follows immediately. $\square$

We can go on and give a very explicit description of flag stabilisers. For that a *flagged set* is a set S together with a sequence $R = (R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n)$ of equivalence relations on S. An *isomorphism* of flagged sets is a bijection of the underlying sets taking the equivalence relations of one set to the equivalence of the other (so that in particular the lengths of the sequences of equivalence relations have to be equal). We denote by $N_R(S)$ the automorphism group of the flagged set. We also use $H \int G$ to denote the wreath product, the semi-direct product $H^n \ltimes G$, where G is a subgroup of Σ_n and operates on H^n by permuting factors accordingly. In our cases H will have a natural embedding in Σ_m and then $H \int G$ embeds naturally in Σ_{mn}.

Proposition 4.2 *Let $(S, R = (R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n))$ be a flagged finite set. On each equivalence class of R_n the equivalence relations $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{n-1}$ induces an equivalence flag. We define an equivalence relation on S by saying that $s \sim t$ if the induced flagged sets $\overline{s}$ and $\overline{t}$ are isomorphic, where $\overline{s}$ and $\overline{t}$ are the equivalence classes in S/R_n containing s resp. t.*

i) $N_R(S)$ is the product over the equivalence classes of this equivalence relation of the flag stabilisers of the flags induced by R on each such equivalence class.

ii) Assume that S consists of a single equivalence class. Then $N_R(S)$ is the wreath product $N_{R'}(S') \int \Sigma_t$. Here t is the number of equivalence classes of R_n, S' is one of those equivalence classes and R' is the equivalence flag of S' induced by $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{n-1}$.

PROOF: This is essentially obvious. First an automorphism of (S,R) will be able to take an equivalence class of R_n to another such equivalence class only if the they are isomorphic as flagged sets. Hence $N_R(S)$ splits as is claimed in i). As for ii) we fix a flagged isomorphism between S' and the other equivalence classes. This allows us to get an action of Σ_t on (S,R). On the other hand it also gives us an isomorphism between $N_{R'}(S')^t$ and the automorphisms of (S,R) taking each element of S/R_n to itself. $\square$

We shall say that a flagged set (possibly with an empty flag, i.e., a flag of length 0) $(S, R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n)$ is a *homogeneous flag* if $R_n = S \times S$ and, provided that $n > 1$, all equivalence classes of R_{n-1} are isomorphic as flagged sets and all the elements of the flag are distinct and different from the least equivalence relation (consisting only of the diagonal). Hence, what the proposition says is that any flag is the disjoint union of flags which become homogeneous after removing repetitions (and possibly adding $S \times S$), that $N_R(S)$ is the corresponding product and that for a homogeneous flag, $N_R(S)$ is an iterated wreath product (which is just a symmetric group in the case of an empty flag).

Theorem 4.3 *We have that $\{\mathrm{B}\Sigma_n\} = 1 \in K_0(\mathrm{Stck}_{\mathbf{k}})$ for all n.*

PROOF: We prove this by induction over n and the case $n = 1$ is trivial. We would like to apply Proposition 3.7 for the standard representation of Σ_n. This is not possible if the characteristic is finite and $\leq n$. However, we can use (1.1:ii) to replace the characteristic dependent arguments of the proposition. The characteristic polynomial for it is $x^n - x$ and as $\mathbb{L}^n - \mathbb{L}$ is invertible in $K_0(\mathrm{Stck}_{\mathbf{k}})$ it is enough to show that $\{\mathrm{B}N_R(S)\} = 1$ for all non-trivial equivalence flags R on S with $|S| = n$ which we shall do by simultaneous induction over the length of the flag and $|S| \leq n$. As $\mathrm{B}(G \times H) = \mathrm{B}(G) \times \mathrm{B}(H)$ we are by Proposition 4.2 reduced to the case when R is homogeneous. Applying again Proposition 4.2 we can write $N_R(S)$ as $N_{R'}(S') \int \Sigma_t$, Furthermore, we have $\mathrm{B}(N_{R'}(S') \int \Sigma_t) = \mathrm{B}(N_{R'}(S')) \int \mathrm{B}\Sigma_t = \mathcal{S}\mathrm{ymm}^t(\mathrm{B}(N_{R'}(S')))$ (where we have somewhat confused constant pseudo-functors and their associated stacks). This gives $\{\mathrm{B}N_R(S)\} = \sigma_s^t(\{\mathrm{B}(N_{R'}(S'))\})$

which by the induction assumption (as R' is shorter than R) is equal to $\sigma_s^t(1) = \{\mathrm{B}\Sigma_t\}$ and as R is non-trivial $t < n$, so by the over-all induction assumption $\{\mathrm{B}\Sigma_t\} = 1$. $\square$

This result has the following somewhat surprising corollary. For it we define, for an algebraic space Y and an algebraic stack X, $\sigma^n_{s,Y}(X) := [X^n \times \mathrm{Conf}^n(Y)/\Sigma_n]$.

Corollary 4.4 *Let X be an algebraic stack and Y an algebraic space of finite type both over* $\mathbf{k}$.

i) We have that

$$\sigma_s^n(\{X\}\{Y\}) = \sum_{\lambda\vdash n}\prod_i \{\sigma^{n_i}_{s,Y}(\sigma_s^{\lambda_i}(X))\},$$

where $\lambda = [\lambda_1^{n_1}, \lambda_2^{n_2}, \dots, \lambda_k^{n_k}]$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ runs over the partitions of n.

ii) There is a universal polynomial (depending only on n) in variables $x_{m,\mu}$ where $\mu = (\mu_1, \dots, \mu_r)$ runs over sequences of positive integers which gives $\sigma^n_{s,Y}(X)$ when evaluated at $x_{m,\mu} = \sigma_s^m(\sigma_s^\mu(\{X\})\{Y\})$ where $\sigma_s^\mu(\{X\}) := \sigma_s^{\mu^1}(\sigma_s^{\mu^2}(\dots(\{X\})))$. In particular $\sigma^n_{s,Y}$ can be naturally extended to $K_0(\mathrm{Spc}_{\mathbf{k}})$.

iii) The two operations λ^n and λ_s^n on $K_0(\mathrm{Stck}_{\mathbf{k}})$ coincide when the characteristic of base field is 0 or $> n$.

Proof: For i) and ii) we follow the proof of Proposition 2.1 (as well as using its notation). Using stack quotients instead of quotients the proof proceeds in the same fashion leading first to i) and then using it and induction gives us ii). Finally, under the extra assumptions on the characteristic we can apply i) and ii) as well as the corresponding results of Proposition 2.1 to get

$$\sigma_s^n(\{X\}) = \sum_{\lambda\vdash n}\prod_i \{\sigma^{n_i}_{s,Y}(\sigma_s^{\lambda_i}(1))\} = \sum_{\lambda\vdash n}\prod_i \{\sigma^{n_i}_{s,Y}(\sigma^{\lambda_i}(1))\} = \sigma^n(\{X\}),$$

where we have used the theorem formulated as $\sigma_s^m(1) = \sigma^m(1) = 1$ for all $m \le n$ and $\sigma^m_{s,Y}(1) = \sigma^m_Y(1)$ as the action of Σ_m on $\mathrm{Conf}^m(Y)$ is free so that $[\mathrm{Conf}^m(Y)/\Sigma_m] = \mathrm{Conf}^m(Y)/\Sigma_m$. $\square$

Remark: It is probably not true that $\lambda^p = \lambda^p_s$ (on $K_0(\mathrm{Spc}_{\mathbf{k}})$ with values in $K_0(\mathrm{Stck}_{\mathbf{k}})$) in characteristic p. Indeed, the same analysis as in the proof of the corollary shows that for a $\mathbf{k}$-scheme X we have $\lambda^p_s(\{X\}) - \lambda^p(\{X\}) = \{X\} - \{\Delta\}$, where Δ is the image of X under the composite $X \xrightarrow{\Delta} X^p \to X^p/\Sigma_p$ and a local computation shows that when X is reduced we have that $\Delta \cong X^{(p)}$ (where $X^{(p)} = X \times_{F_{\mathbf{k}}} \mathbf{Spec}\,\mathbf{k}$). Unless X is defined over $\mathbf{F}_p$, X and $X^{(p)}$ are in general not isomorphic. For X an elliptic curve with j-invariant not in $\mathbf{F}_p$ we should probably expect $\{X\} \ne \{X^{(p)}\}$ in $K_0(\mathrm{Stck}_{\mathbf{k}})$. (Note that if we have resolution of singularities then an abelian variety can be recovered from its class in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$.)

5 Non-triviality results

We are now going to obtain some results on the non-triviality of $\{\mathrm{B}G\}$ for G a finite group. All these results require at their basis resolution of singularities so *in this section we shall assume that* $\mathbf{k}$ *has characteristic zero*. In [Ek09a, Prop. 3.3] a Grothendieck group $L_0(\mathrm{Ab})$ of finitely generated abelian groups with only the relations $\{A \bigoplus B\} = \{A\} + \{B\}$ was introduced. Thus $L_0(\mathrm{Ab})$ is the free abelian group on the indecomposable finitely generated abelian groups $\{\mathbf{Z}\}$ and $\{\mathbf{Z}/p^n\}$ for primes p and $n > 0$. For each integer k a continuous homomorphism $H^k(-)\colon \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to L_0(\mathrm{Ab})$ was defined characterised by the property that $H^k(\{X\}/L^m) = \{H^{k+2m}(X, \mathbf{Z})\}$, where X is smooth and proper.

Remark: This as it stands makes sense only when $\mathbf{k} = \mathbf{C}$ as only then does $H^k(X, \mathbf{Z})$ exist. Otherwise, we can still define $\{H^k(X, \mathbf{Z})\}$ as $b_k(X)\{\mathbf{Z}\} + \sum_p \{\mathrm{tor}(H^k(X, \mathbf{Z}_p))\}$.

We define $L_0^f(\mathrm{Ab})$ as the subgroup of $L_0(\mathrm{Ab})$ spanned by the classes of finite groups. In general any element of $L_0(\mathrm{Ab})$ can be written uniquely as $m\{\mathbf{Z}\} + a$ where $a \in L_0^f(\mathrm{Ab})$ and we call a the *torsion part* of the element and $m\{\mathbf{Z}\}$ its *torsion free part* and that the element is *torsion free* if $a = 0$ and *torsion* of $m = 0$.

Theorem 5.1 *Let G be a finite group. Then*

- $H^k(\{\mathrm{B}G\}) = 0$ *for* $k > 0$,
- $H^0(\{\mathrm{B}G\}) = \{\mathbf{Z}\}$,
- $H^{-1}(\{\mathrm{B}G\}) = 0$,
- $H^{-2}(\{\mathrm{B}G\}) = \{B_0(G)^\vee\}$, *where $B_0(G)$ is the subgroup of $H^2(G, \mathbf{Q}/\mathbf{Z})$ consisting of those elements that map to zero upon restriction to all abelian subgroups of G (and $B_0(G)^\vee$ its dual as a finite group) and*
- $H^k(\{\mathrm{B}G\}) \in L_0^f(\mathrm{Ab})$ *for* $k < 0$.

PROOF: Choose a faithful linear representation V of G of dimension n. By the continuity of H^k and by Proposition 3.1 we have that for large m (the size depending on k) $H^k(\{\mathrm{B}G\}) = \{H^{k+2nm}(\{V^m/G\})\}$. Using compactification and resolution of singularities we may write $\{V^m/G\}$ as a linear combination $\sum_i n_i\{X_i\}$ of classes of smooth and proper varieties all of dimension $\leq \dim(V^m/G) = 2nm$ and then $H^k(\{\mathrm{B}G\}) = \sum_i\{H^{k+2nm}(X_i, \mathbf{Z})\}$. As $\dim X_i \leq 2nm$ for all i we immediately get that $H^k(\{\mathrm{B}G\}) = 0$ for $k > 0$. Let X be a smooth and proper variety birational to V^m/G. We can then be more precise and write $\{V^m/G\}$ as $\{X\}+\sum_i n'_i\{X'_i\}$, where the X'_i are smooth and proper of dimensions $< 2mn$. Hence $\{H^{2nm}(X'_i, \mathbf{Z})\} = \{H^{2nm-1}(X'_i, \mathbf{Z})\} = 0$ and consequently $H^0(\{\mathrm{B}G\}) = \{H^{2nm}(X, \mathbf{Z})\} = \{\mathbf{Z}\}$ and $H^{-1}(\{\mathrm{B}G\}) = \{H^{2nm-1}(X, \mathbf{Z})\}$. Now, X being unirational is simply-connected and by Poincaré duality $\{H^{2nm-1}(X, \mathbf{Z})\} = \{H_1(X, \mathbf{Z})\} = 0$. Similarly, $\{H^{2nm-2}(X'_i, \mathbf{Z})\}$ is torsion free and hence the torsion part of $H^{-2}(\{\mathrm{B}G\})$ is equal to the torsion part of $\{H^{2nm-2}(X, \mathbf{Z})\}$. Now by Poincaré duality the torsion of $\{H^{2nm-2}(X, \mathbf{Z})\}$ is dual to the torsion of $\{H^3(X, \mathbf{Z})\}$ which is the (cohomological) Brauer group of X modulo its divisible part. However, as X is unirational the divisible part is zero so that the torsion of $\{H^3(X, \mathbf{Z})\}$ is the Brauer group. This is a birational invariant and defined directly in terms of the invariant field $\mathbf{k}(V^m)^G$ as the unramified Brauer group. This group is isomorphic to $B_0(G)$ (cf., [Bo97, Thm. 3.1]).

Thus what remains to be shown is that $H^k(\{\mathrm{B}G\}) \in L_0^f(\mathrm{Ab})$ for $k \neq 0$. Recall (cf., [Ek09a, Prop. 3.2]) that if we compose $H^k(-)$ with the map $-\bigotimes\mathbf{Q}\colon L_0(\mathrm{Ab}) \to L_0(\mathbf{Q}\text{−vec})$ given by $\{A\}\bigotimes\mathbf{Q} = \{A\bigotimes\mathbf{Q}\}$ (and $L_0(\mathbf{Q}\text{−vec})$ is the Grothendieck group of finite dimensional $\mathbf{Q}$-vector spaces) then for $x \in \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ we have that $(-1)^k H^k(x)\bigotimes\mathbf{Q}$ is the weight k-part of $\chi_c(x)$. (More precisely, we have a group homomorphism $K_0(\mathrm{Coh}_{\mathbf{k}}) \to L_0(\mathbf{Q}\text{−vec})$ taking $\{V\}$ to $\dim V\{\mathbf{Q}\}$ and $(-1)^k H^k(x)\bigotimes\mathbf{Q}$ is the image of the weight k-part of $\chi_c(x)$ under this map.) Now, we have (by Proposition 3.1) that $\chi_c(\{\mathrm{B}G\}) = 1$ and hence is pure of weight 0. □

Corollary 5.2 *There are finite groups G for which $\{\mathrm{B}G\} \neq 1$ in $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ for all fields $\mathbf{k}$.*

PROOF: It is enough, by the theorem, to find finite groups G for which $B_0(G) \neq 0$. The first such examples were given by Saltman, [Sa84, Thm. 3.6], and were groups of order p^9 for any prime p (using the definition of $B_0(G)$ as the unramified Brauer group). Bogomolov, [Bo97], then obtained the group-cohomological description of $B_0(G)$ and found examples of order p^6 with non-trivial $B_0(G)$, those orders are minimal in terms of divisibility. □

5.0.1 Abelian étale groups

We are now going to see if we can go backwards in our previous arguments concerning $\{\mathrm{B}\mathbf{Z}/p\}$ to detect non-triviality (still in characteristic zero of course). We can start with a somewhat more general situation: We thus consider a finite étale $\mathbf{k}$-group scheme A and an embedding av A into a torus T. Provided that $T \in \mathcal{Z}\mathrm{ar}_{\mathbf{k}}$ we then have $\{\mathrm{B}A\} = \{T/A\}/\{T\}$ so we need to obtain some information on $\{U\}$ for a torus U. Using [CHS05] we can find a (projective) toric compactification X of U (defined over $\mathbf{k}$). In particular the complement of U in X is a divisor with

normal crossings and we denote by $\{X_s\}_{s\in S}$ its geometric irreducible components (which are thus permuted by the Galois group of $\mathbf{k}$). The fact that the Galois action in general will be non-trivial makes for instance the inclusion-exclusion formulas a little bit tricky and we start by introducing a formalism that will make them easier to handle. Hence, we recall (cf., [Bi04]) that one defines the Grothendieck group $K_0(\mathrm{Spc}_X)$ of schemes over a $\mathbf{k}$-scheme X as generated by isomorphism classes $\{Y \to X\}$ of schemes over X with relations $\{Y \to X\} = \{Y \setminus Y' \to X\} + \{Y' \to X\}$ for a closed subscheme $Y' \subseteq Y$ (we are only going to use this for X a finite étale $\mathbf{k}$-scheme). If $f\colon X' \to X$ is a morphism we have a pushforward map $f_*\colon K_0(\mathrm{Spc}_{X'}) \to K_0(\mathrm{Spc}_X)$ given by composition with f and a pullback map $f^*\colon K_0(\mathrm{Spc}_X) \to K_0(\mathrm{Spc}_{X'})$ given by pullback. The situations we are going to encounter deal with the case when $X \to \mathbf{Spec}\,\mathbf{k}$ is finite étale and X' is constructed from X by making some extra choices. The push forward map will the be written as a sum over the extra choices. This notation fulfils the expected formulas, for instance:

- An iterated sum can be replaced a single sum over all the choices involved, this is transitivity $(fg)_* = f_*g_*$ of pushforward.
- A term not depending on the extra choices may be moved out of the sum, this is the projection formula.

As a first example, assume that $S \to \mathbf{Spec}\,\mathbf{k}$ is finite étale of rank n and let $\mathcal{P}(S) \to \mathbf{Spec}\,\mathbf{k}$ be the finite étale map of flags $T_1 \subset T_2 \subset \cdots \subset T_k \subset S$ (of varying length) of sub-$\mathbf{k}$-schemes of S. Then we put

$$\mathrm{sign}_S := (-1)^n \sum_{T_1\subset\cdots\subset T_{k-1}\subset S} (-1)^k.$$

This means more explicitly that we first define an element $(-1)^k \in K_0(\mathrm{Spc}_{\mathcal{P}(S)})$ by writing $\mathcal{P}(S)$ as the disjoint union $\mathcal{P}^k(S)$ of flags of length k. We then have pushforward maps induced by the inclusions $\mathcal{P}^k(S) \subseteq \mathcal{P}(S)$ and we define $(-1)^k$ as the sum over k of the pushforwards of $(-1)^k \in K_0(\mathrm{Spc}_{\mathcal{P}^k(S)})$. This procedure can be written as

$$\sum_{T_1\subset\cdots\subset T_{k-1}\subset S} (-1)^k = \sum_k \sum_{T_1\subset\cdots\subset T_{k-1}\subset S} (-1)^k = \sum_k (-1)^k \sum_{T_1\subset\cdots\subset T_{k-1}\subset S} 1.$$

With this formalism we get an inclusion-exclusion formula which looks very much like the usual one.

Proposition 5.3 *Let X be a $\mathbf{k}$-scheme, $S \to \mathbf{Spec}\,\mathbf{k}$ a finite étale map and X' a closed subscheme of $S \times X$. Let U be the complement of the union $\cup_{s\in S} X_s$, where s runs over $\bar{\mathbf{k}}$-points of S and X_s is the fibre over s of X', then U is an open $\mathbf{k}$-subscheme of X. Let $\mathcal{P}(S) \to \mathbf{k}$ be the étale $\mathbf{k}$-scheme of subschemes of S with universal subscheme $T \subseteq \mathcal{P}(S) \times S$. If we put $X_T := \cap_{s\in T} X_s$, a closed subscheme of $\mathcal{P}(S) \times X$, we have the formula*

$$\{U\} = \sum_{T\subseteq S} (-1)^{|T|} \mathrm{sign}_T \{X_T\} \in K_0(\mathrm{Spc}_{\mathbf{k}}).$$

Proof: Let U_T be the complement in X_T of the union of the $X_{T'}$ for $T \subset T' \subseteq S$. Then X_T is the disjoint union of the $U_{T'}$ for $T \subseteq T' \subseteq S$ so that we have

$$\{X_T\} = \sum_{T\subseteq T'\subseteq S} \{U_{T'}\} = \{U_T\} + \sum_{T\subset T'\subseteq S} \{U_{T'}\} \in K_0(\mathrm{Spc}_{\mathcal{P}(S)}).$$

The rest is then essentially to apply Möbius inversion for this formula taking care to keep track of pushforwards. The summation formalism takes care of that however. Thus iterating the above

formula gives

$$\{U_T\} = \{X_T\} - \sum_{T\subset T_1}\{U_{T_1}\} = \{X_T\} - \sum_{T\subset T_1}\{X_{T_1}\} + \sum_{T\subset T_1\subset T_2}\{U_{T_2}\} = \cdots =$$
$$\sum_{T\subset T_1\subset\cdots\subset T_k}(-1)^k\{X_{T_k}\} = \sum_{T\subseteq T'}\{X_{T'}\}\sum_{T\subset T_1\subset\cdots\subset T_{k-1}\subset T'}(-1)^k = \sum_{T\subseteq T'}(-1)^{|T'\setminus T|}\mathrm{sign}_{T'\setminus T}\{X_{T'}\}$$

which applied to $T=\emptyset$ gives the proposition. □

Remark: i) In [Rö07] it is shown that if S is of $\mathbf{k}$-rank n, then sign_S is equal to $(-1)^n\lambda^n(\{S\})$. We are not going to use that but only the same formula in a Grothendieck ring of Galois representations where we shall provide a shorter proof. In that case $\lambda^n(\{\mathbf{Z}[S]\})$ is the signum character which seems conceptually reasonable.

ii) The kind of general Möbius inversion formula obtained in the proposition will be studied more systematically elsewhere.

We recall ([Ek09a, Thm. 3.4]) that we have homomorphisms $\mathrm{NS}^k\colon \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to L_0(\mathrm{Ab}_{\mathbf{k}})$, where $L_0(\mathrm{Ab}_{\mathbf{k}})$ is the Grothendieck group of isomorphism classes of étale $\mathbf{k}$-group schemes geometrically finitely generated and relations $\{A\bigoplus B\}=\{A\}+\{B\}$. We shall denote by $K_0(\mathrm{Ab}_{\mathbf{k}})$ the Grothendieck group of the abelian category of continuous actions of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ on finitely generated abelian groups. We then have a continuous group homomorphism $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}}) \to K_0(\mathrm{Ab}_{\mathbf{k}})$ which is the composite of NS and the natural map $L_0(\mathrm{Ab}_{\mathbf{k}}) \to K_0(\mathrm{Ab}_{\mathbf{k}})$ (taking $\{A\}$ to $\{A\}$). Unsurprisingly $L_0(\mathrm{Ab}_{\mathbf{k}})$ contains more information than $K_0(\mathrm{Ab}_{\mathbf{k}})$. One way of extracting more information is to consider a continuous finite quotient $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \to G$ with kernel N and then regard the maps $(-)^N,(-)_N\colon L_0(\mathrm{Ab}_{\mathbf{k}}) \to L_0(\mathrm{Ab}_G)$, where $L_0(\mathrm{Ab}_G)$ is generated by isomorphism class of finitely generated G-modules with the sum relation as above, taking $\{A\}$ to the (co)invariants under N $\{A^N\}$ (resp. $\{A_N\}$). (Note that while taking (co)invariants is not exact it is additive which is enough to give the induced maps.) We can then compose with the natural map $L_0(\mathrm{Ab}_G) \to K_0(\mathrm{Ab}_G)$ (with the obvious meaning of $K_0(\mathrm{Ab}_G)$) to get maps $(-)^N,(-)_N\colon L_0(\mathrm{Ab}_{\mathbf{k}}) \to K_0(\mathrm{Ab}_G)$. Note that we have a ring structure on any $K_0(\mathrm{Ab}_?)$ or $L_0(\mathrm{Ab}_?)$ characterised by $\{A\}\{B\}=\{A\bigotimes B\}$ when either of A and B are torsion free and similarly a λ-ring structure (special on $K_0(\mathrm{Ab}_?)$) characterised by $\lambda^i(\{A\})=\{\Lambda^i(A)\}$ when A is torsion free.

In order to get an inclusion-exclusion formula we need to extend NS even further. Hence we define, for a finite étale $\mathbf{k}$-scheme, $S\to\mathbf{Spec}\,\mathbf{k}$, $L_0(\mathrm{Ab}_S)$ resp. $K_0(\mathrm{Ab}_S)$ as the Grothendieck group of étale S-group schemes (or equivalently locally constant sheaves) geometrically finitely generated (resp. finitely generated and torsion free) and relations $\{A\bigoplus B\}=\{A\}+\{B\}$ (resp. $\{C\}=\{A\}+\{B\}$ for short exact sequences $0\to A\longrightarrow C\longrightarrow B\to 0$). The maps Num^k and NS^k then extend to this situation. For a map $f\colon S\to T$ of finite étale $\mathbf{k}$-schemes the direct and inverse images induce maps $f_*\colon L_0(\mathrm{Ab}_S)\to L_0(\mathrm{Ab}_T)$, $f_*\colon K_0(\mathrm{Ab}_S)\to K_0(\mathrm{Ab}_T)$, $f^*\colon L_0(\mathrm{Ab}_T)\to L_0(\mathrm{Ab}_S)$ and $f^*\colon K_0(\mathrm{Ab}_T)\to K_0(\mathrm{Ab}_S)$. Furthermore, the expected relations hold; transitivity $(fg)_*=g_*f_*$ and $(fg)^*=g^*f^*$ and the projection formula $xf_*y=f_*(f^*xy)$. Because of these relations we may (and shall) use the summation formalism also for $L_0(\mathrm{Ab}_-)$ and $K_0(\mathrm{Ab}_-)$.

Lemma 5.4 *i) Pushforwards and pullbacks commute with* NS^k *and* Num^k.

ii) If $x\in K_0(\mathrm{Spc}_{\mathbf{k}})$ *is in the subgroup generated by classes of zero-dimensional schemes, then* $\mathrm{NS}^k(xy)=\mathrm{NS}^0(x)\mathrm{NS}^k(y)$ *for all* $y\in K_0(\mathrm{Spc}_{\mathbf{k}})$.

iii) If S *is a finite étale* $\mathbf{k}$*-scheme of rank* n, *then* $\mathrm{Num}^0(\mathrm{sign}_S)=\{\det\mathbf{Z}[S]\}$ *and* $\mathrm{Num}^k(\mathrm{sign}_S)=0$ *for* $k\neq 0$.

Proof: The first part is clear as if $\mathbf{k}$ is algebraically closed, pushforward and pullback are expressed in terms of disjoint unions resp. direct sums, $\mathrm{CH}^*(-)$ and $H^*(-,\widehat{\mathbf{Z}})$ takes disjoint unions to direct sums and the Galois actions are compatible.

To prove the second part we may reduce to the case when $x = \{S\}$ and in that case $xy = f_*f^*y$, where $f\colon S \to \mathbf{Spec}\,\mathbf{k}$ is the structure map. Thus the first part shows that $\mathrm{NS}^k(xy) = f_*f^*\mathrm{NS}^k(y) = f_*1 \cdot \mathrm{NS}^k(y)$ and as $1 = \mathrm{NS}^0(1)$ we conclude by another application of the first part.

As for the last part, that $\mathrm{Num}^k(\mathrm{sign}_S) = 0$ for $k \neq 0$ is clear and we have, by the first part and the fact that $\mathrm{Num}^0(1) = 1 \in K_0(\mathrm{Ab}_U)$ for any étale $\mathbf{k}$-scheme U,

$$\mathrm{Num}^0(\mathrm{sign}_S) = (-1)^n \sum_{T_1 \subset \cdots \subset T_{k-1} \subset S} (-1)^k \mathrm{Num}^0(1) = (-1)^n \sum_{T_1 \subset \cdots \subset T_{k-1} \subset S} (-1)^k.$$

Now the sum runs over the simplices of the barycentric subdivision of the boundary of the n-simplex, where we count also the empty simplex. However, there is an extra sign so the sum is minus the equivariant Euler characteristic of the reduced homology of the n-sphere. This homology is just $\mathbf{Z}$ in degree $n-1$ where the symmetric group acts by the signature character. □

In general there is no particular relation between these lambda ring structures, the NS^k and the lambda ring structure on $K_0(\mathrm{Spc}_{\mathbf{k}})$ but something can be said for special elements. Hence we define $\mathrm{ArtL}_{\mathbf{k}} \subset \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ as the closure of the ring of finite sums $\sum_i a_i \mathbb{L}^i$ where the a_i are linear integral combinations of classes of zero-dimensional schemes.

Definition-Lemma 5.5 *For $x \in \widehat{K}_0(\mathrm{Spc}_x)$ put $\mathrm{NS}_s(x) := \sum_k \mathrm{NS}^k(x)s^k \in L_0(\mathrm{Ab}_{\mathbf{k}})((s^{-1}))$.*

i) If $x \in \mathrm{ArtL}_{\mathbf{k}}$ and $y \in \widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})$ then $\mathrm{NS}_s(xy) = \mathrm{NS}_s(x)\mathrm{NS}_s(y)$.

ii) If we give $L_0(\mathrm{Ab}_{\mathbf{k}})((s^{-1}))$ a lambda ring structure by requiring the λ^i to be continuous and $\lambda^i(xs^j) = \lambda^i(x)s^{ij}$ then for $x \in \mathrm{ArtL}_{\mathbf{k}}$ we have $\mathrm{NS}_s(\lambda^i(x)) = \lambda^i(\mathrm{NS}_s(x))$.

Proof: By continuity and additivity of the NS^k for the first part we are reduced to the case when $x = a\mathbb{L}^i$ and $y = b\mathbb{L}^j$ with $a \in \mathrm{ArtL}_{\mathbf{k}}$ and $b \in K_0(\mathrm{Spc}_{\mathbf{k}})$ in which case it follows from the definition of NS^k and Lemma 5.4.

For the second part assume first that the statement is true for x a linear integral combination of classes of zero-dimensional schemes. If $x = \sum_i a_i \mathbb{L}^i$ we have (where we extend NS_s to $\widehat{K}_0(\mathrm{Spc}_{\mathbf{k}})[[t]]$ by $\mathrm{NS}_s(\sum_i x_i t^i) = \sum_i \mathrm{NS}_s(x_i)t^i$), using that Num_s is a ring homomorphism on ArtL (which follows from the first part)

$$\mathrm{NS}_s(\lambda_t(x)) = \mathrm{NS}_s(\lambda_t(\sum_i a_i \mathbb{L}^i)) = \mathrm{NS}_s(\prod_i \lambda_{\mathbb{L}^i t}(a_i)) = \\ \prod_i \lambda_{s^i t}(\mathrm{NS}_s(a_i)) = \lambda_t(\sum_i \mathrm{NS}_s(a_i)s^i) = \lambda_t(\mathrm{NS}_s(x)).$$

What then remains to be shown is that $\mathrm{NS}^0(\lambda^i(\{S\})) = \lambda^i(\mathrm{NS}^0(\{S\}))$ for a finite étale $\mathbf{k}$-scheme S. As we already know that NS^0 is a ring homomorphism on the subgroup spanned by such classes it is enough to show the same formula for σ^i instead of λ^i. In that case $\sigma^i(\{S\}) = \{S^i/\Sigma^i\}$, $\mathrm{NS}^0(\{S\}) = \mathbf{Z}[S]$, $\mathrm{NS}^0(\sigma^i(\{S\})) = \mathbf{Z}[S^i/\Sigma_i]$ and $\sigma^i(\{\mathbf{Z}[S]\}) = \{S^i\mathbf{Z}[S]\}$ and we have a natural isomorphism $S^i\mathbf{Z}[S] = \mathbf{Z}[S^i/\Sigma_i]$. □

Theorem 5.6 *Let U be an n-dimensional $\mathbf{k}$-torus with cocharacter group M considered as a $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-module.*

i) $\mathrm{NS}^k(\{U\})_K = (-1)^{n-k}\{\Lambda^{n-k}(M)\} \in K_0(\mathrm{Ab}_G)$, where G is the image of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ in $\mathrm{Aut}(M)$ and K is the kernel of the map $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$.

ii) If Δ is an étale $\mathbf{k}$-sheaf of fans in M which give a smooth and proper toric compactification of U, S the sheaf of rays of Δ and $f\colon \mathbf{Z}[S] \to M$ the surjective map induced from the inclusion $S \subset M$ then $\mathrm{NS}^{n-1}(U) = \{\ker f\} - \{\mathbf{Z}[S]\} \in L_0(\mathrm{Ab}_{\mathbf{k}})$.

Proof: Let N be the character group of U (which is the dual of M). It is a fact (cf., [Fu93, Thm. p. 102]) that over $\bar{\mathbf{k}}$ the cycle class map $\mathrm{CH}^k(Y) \bigotimes \mathbf{Z}_p \to H^{2k}(Y, \mathbf{Z}_p)$ is an isomorphism for

any smooth and projective toric variety Y. This means that we have $\mathrm{NS}^k(\{Y\}) = \{\mathrm{CH}^k(Y)\} \in L_0(\mathrm{Ab}_{\mathbf{k}})$. Here we let $\mathrm{CH}^k(Y)$ be the étale group scheme with $\mathrm{CH}^k(Y)(\bar{\mathbf{k}}) = \mathrm{CH}^k(Y_{\bar{\mathbf{k}}})$ with its natural Galois action.

For the first part we shall work exclusively with G-modules and we shall suppress $(-)_K$, note that we have $\mathrm{CH}^k(X)_K = \mathrm{CH}^k(X)$ and similarly for $H^{2k}(-,\widehat{\mathbf{Z}})$. Now, given a smooth and proper toric $\mathbf{k}$-variety Y corresponding to a character group N with a Galois action and a fan (we think of a fan as its set of faces) Δ in the dual of N invariant under the Galois group we have (cf., [Da78, §10]) that the Chow ring of $Y_{\bar{\mathbf{k}}}$ is isomorphic to $\mathrm{SR}_\Delta/(N)$, where SR_Δ is the Stanley-Reisner ring of Δ and the map $N \to \mathrm{SR}_\Delta$ is defined as follows: By definition we have a map $\mathbf{Z}[S] \to \mathrm{SR}_\Delta$ into the degree 1-part, where S is the set of rays of the fan Δ. The elements of N give functions of S by evaluating $n \in N$ on the integral generator of a ray. This gives a map $N \to \mathbf{Z}[S]^* = \mathbf{Z}[S]$. All these maps respect the action of the Galois group. Furthermore, a basis for N give a regular sequence in SR_Δ so that the Koszul complex for $N \to \mathrm{SR}_\Delta$ is exact. We can then perform the standard computation of the Hilbert series of $\mathrm{SR}_\Delta/(N)$, where one splits up the monomial basis of it by its support, a face of Δ. This gives the following equalities of $K_0(\mathrm{Ab}_G)$-valued Hilbert series

$$\mathrm{Hilb}(\mathrm{CH}(Y)) = \lambda_{-s}(\{N\})\mathrm{Hilb}(\mathrm{SR}_\Delta) = \lambda_{-s}(\{N\})\sum_{T\in\Delta} s^{|T|}\sigma_s(\{\mathbf{Z}[T]\}), \tag{4}$$

where T runs over the faces of Δ (and as before the summation is interpreted as a pushforward from the étale $\mathbf{k}$-scheme of faces of Δ, where Δ is the étale scheme of faces of Δ, or equivalently U-orbits on Y).

We want to combine this with Proposition 5.3 and as the intersection of the divisors is empty when the index set of the intersection is not a face we do not have to sum over all subsets, only faces. Suppose thus that T is a face. Then the intersection Y_T is a smooth toric variety whose torus has character group N_T which is the kernel of the surjective evaluation map $N \to \mathbf{Z}[T]$ and hence using (4) and standard formulas for lambda rings

$$\begin{aligned}\mathrm{Hilb}(\mathrm{CH}(Y_T)) = \lambda_{-s}(\{N_T\})\sum_{T\subseteq T'\in\Delta} s^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T'\setminus T]\}) = \\ \lambda_{-s}(\{N\})\sigma_s(\{\mathbf{Z}[T]\})\sum_{T\subseteq T'\in\Delta} s^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T'\setminus T]\}) = \lambda_{-s}(N)\sum_{T\subseteq T'\in\Delta} s^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T']\}).\end{aligned}$$

Using Proposition 5.3 we get (where we use sign_T also for $\mathrm{NS}^0(\mathrm{sign}_T)$)

$$\begin{aligned}\mathrm{NS}_s(U) = \lambda_{-s}(\{N\})\sum_{T\subseteq T'\in\Delta}(-1)^{|T|}\mathrm{sign}_T s^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T']\}) = \\ \lambda_{-s}(\{N\})\sum_{T\subseteq T'\in\Delta}(-1)^{|T|}\mathrm{sign}_T t^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T']\}).\end{aligned}$$

The sum may then be rewritten as

$$\begin{aligned}\sum_{T\subseteq T'\in\Delta}(-1)^{|T|}\mathrm{sign}_T s^{|T'\setminus T|}\sigma_s(\{\mathbf{Z}[T']\}) = \\ \sum_{T'\in\Delta}(-1)^{|T'|}\mathrm{sign}_{T'}\sigma_s(\{\mathbf{Z}[T']\})\sum_{T\subseteq T'}\mathrm{sign}_{T'\setminus T}(-s)^{|T'\setminus T|},\end{aligned}$$

where we have used that $\mathrm{sign}_{T'} = \mathrm{sign}_{T'}\mathrm{sign}_{T'\setminus T}$ and $\mathrm{sign}^2_{T'} = 1$. Now,

$$\sum_{T\subseteq T'}\mathrm{sign}_{T'\setminus T}(-s)^{|T'\setminus T|} = \sum_{0\le k\le m}(-s)^{m-k}\sum_{\substack{T\subseteq T'\\ |T|=m-k}}\mathrm{sign}_T = \sum_{0\le k\le m}(-s)^k\sum_{\substack{T\subseteq T'\\ |T|=k}}\mathrm{sign}_T,$$

where $m := |T'|$, and the inner sum is just $\{\Lambda^k \mathbf{Z}[T']\}$ so that the full sum is equal to $\lambda_{-s}(\{\mathbf{Z}[T']\})$. This gives

$$\mathrm{Num}_s(U) = \lambda_{-s}(\{N\}) \sum_{T' \in \Delta} (-1)^{|T'|} \mathrm{sign}_{T'} \sigma_s(\{\mathbf{Z}[T']\}) \lambda_{-s}(\{\mathbf{Z}[T']\}) =$$

$$\lambda_{-s}(\{N\}) \sum_{T' \in \Delta} (-1)^{|T'|} \mathrm{sign}_{T'}.$$

The sum is the reduced Euler characteristic of Δ (again with an extra sign) considered as a simplicial complex (it is unoriented which is the reason why one has to insert the $\mathrm{sign}_{T'}$). Now, Δ is a triangulation of the sphere of directions of $M \bigotimes \mathbf{R}$ so that its reduced homology is concentrated in degree $n-1$ and is $\mathbf{Z}$ there. In general if V is an n-dimensional real vector space and S its sphere of directions (i.e., $V \setminus \{0\}$ modulo rays) then there is a canonical identification $H_{n-1}(S, \mathbf{R}) = \Lambda^n V$. Hence, in our case the Galois action on $H_{n-1}(\Delta, \mathbf{R})$ is the same as the action on $\Lambda^n(M) \bigotimes \mathbf{R}$. However, as the action on both $H_{n-1}(\Delta, \mathbf{Z})$ and $\Lambda^n(M)$ are given by characters this implies that $H_{n-1}(\Delta, \mathbf{Z}) \cong \Lambda^n(M)$ and thus

$$\sum_{T' \in \Delta} (-1)^{T'} \mathrm{sign}_{T'} = (-1)^n \lambda^n(\{M\})$$

giving

$$\mathrm{Num}_s(U) = \lambda_{-s}(\{N\}) \lambda^n(\{M\})(-1)^n$$

and separating powers of t gives

$$\mathrm{Num}^k(\{U\}) = (-1)^{n+k-1} \{\Lambda^k N \bigotimes \Lambda^n M\}.$$

Now, the wedge product gives isomorphisms $\Lambda^{n-k} M = (\Lambda^k M)^\vee \bigotimes \Lambda^n M$ and as we also have natural isomorphisms $(\Lambda^k M)^\vee = \Lambda^k M^\vee = \Lambda^k N$ we get

$$\mathrm{Num}^k(\{U\}) = (-1)^{n+k} \{\Lambda^{n-k}(M)\} = (-1)^{n-k} \{\Lambda^{n-k}(M)\}$$

which proves the first part.

As for the second part we choose again a toric compactification X of U and apply NS^k to the formula of Proposition 5.3 for $k = n-1$. Using that $\mathrm{CH}^{n-1}(Y) = 0$ if $\dim Y < n-1$ and $\mathrm{CH}^{n-1}(Y) = \mathbf{Z}[S]$ if $\dim Y = n-1$ and S the $\mathbf{k}$-scheme of $n-1$-dimensional irreducible geometric components we get that

$$\mathrm{NS}^{n-1}(\{U\}) = \{\mathrm{CH}^{n-1}(X)\} - \{\mathbf{Z}[S]\},$$

where S is the scheme of irreducible geometric components of $X \setminus U$ (or equivalently the rays of the fan Δ of X). Now, multiplication in the Chow ring gives a perfect duality (as the Chow groups tensored with $\widehat{\mathbf{Z}}$ is the cohomology of X) between $\mathrm{CH}^{n-1}(X)$ and $\mathrm{CH}^1(X) = \mathrm{Pic}(X)$. As a split torus has trivial Picard group we get an exact sequence

$$0 \to N \longrightarrow \mathbf{Z}[S] \longrightarrow \mathrm{Pic}(X) \to 0$$

and dualising gives an exact sequence

$$0 \to \mathrm{CH}^{n-1}(X) \longrightarrow \mathbf{Z}[S] \longrightarrow M \to 0$$

which finishes the proof. □

To be able to formulate the next theorem let us introduce an involution $(-)^\vee$ of $K_0(\mathrm{Ab}_?)$ given by $\{A\}^\vee = \{\mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})\} - \{\mathrm{Ext}^1_{\mathbf{Z}}(A, \mathbf{Z})\}$. In particular, if A is finite we have that $\{A\}^\vee = -\{\mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})\}$, i.e., minus the usual dual of A.

Theorem 5.7 *Let A be a finite étale commutative* $\mathbf{k}$*-group scheme. If G is the image of* $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ *in* $\mathrm{Aut}(A)$ *then* $\mathrm{NS}_s(\{\mathrm{B}A\}) = \lambda_{-s^{-1}}(\{A\}) \in K_0(\mathrm{Ab}_G)$.

Proof: We shall consistently confuse étale commutative $\mathbf{k}$-group schemes with representations of $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ (and generally étale k-schemes with $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-sets). Further, as we shall only consider such sets for which the action factors through G we shall also confuse with G-representations (G-sets). Pick a $\mathbf{k}$-subscheme S of A' such that it generates A' (we may choose $S = A'$). This gives us a surjective homomorphism of étale sheaves $N := \mathbf{Z}[S] \to A'$ and let N' be its kernel. Let T resp. T' be the $\mathbf{k}$-tori corresponding to N resp. N'. In particular T is the torus of units in $\Gamma(S, \mathcal{O}_S)$ so that $T \in \mathcal{Z}\mathrm{ar}$. The exact sequence $0 \to N' \longrightarrow N \longrightarrow A' \to 0$ gives an exact sequence of group schemes $0 \to A \longrightarrow T \longrightarrow T' \to 0$ and hence, as $T \in \mathcal{Z}\mathrm{ar}$, we have $\{\mathrm{B}A\} = \{T'\}\{T\}^{-1}$. If we let M resp. M' be the cocharacter groups of T resp. T' (i.e., the duals of N resp. N') we get from Theorem 5.6 that $\mathrm{NS}_s(\{T\}) = s^n\lambda_{-s^{-1}}(\{M\})$ (resp. $\mathrm{NS}_s(\{T'\}) = s^n\lambda_{-s^{-1}}(\{M'\})$) where $n := \dim T = \dim T'$. In [Rö07] a formula for $\{T\} \in K_0(\mathrm{Spc}_{\mathbf{k}})$ is proven which implies that $\{T\} \in \mathrm{ArtL}_{\mathbf{k}}$. Hence by Lemma 5.5 we have

$$\mathrm{NS}_s(\{\mathrm{B}A\}) = \mathrm{NS}_s(\{T'\})\mathrm{NS}_s(\{T\})^{-1} = \lambda_{-s^{-1}}(\{M'\})\lambda_{-s^{-1}}(\{M\})^{-1}$$

and as dualising the exact sequence $0 \to N' \longrightarrow N \longrightarrow A' \to 0$ gives us an exact sequence $0 \to M \longrightarrow M' \longrightarrow A \to 0$ we have

$$\lambda_{-s^{-1}}(\{M'\})\lambda_{-s^{-1}}(\{M\})^{-1} = \lambda_{-s^{-1}}(\{M'\} - \{M\}) = \lambda_{-s^{-1}}(\{A\}).$$

□

From this theorem it follows that there are many étale group schemes A (over suitable fields $\mathbf{k}$) for which $\{\mathrm{B}A\} \neq 1$. In fact, by the localisation sequence of algebraic K-theory, for any finite group G the kernel of the map $K_0(\mathrm{Ab}_{\mathbf{Z}[G]}) \to K_0(\mathrm{Ab}_{\mathbf{Q}[G]})$ is generated by classes of torsion G-modules and this kernel is often non-trivial. Then for any torsion G-module A whose class $\{A\} \in K_0(\mathrm{Ab}_G)$ is non-trivial and any field $\mathbf{k}$ whose Galois group has G as a quotient we get an étale group scheme A for which $\{\mathrm{B}A\}$ is different from 1. We content ourselves by noting that Swan's example does indeed give a non-triviality result also in our context.

Corollary 5.8 $\{\mathrm{B}\mathbf{Z}/47\mathbf{Z}\} \neq 1$ *in* $\widehat{K}_0(\mathrm{Spc}_{\mathbf{Q}})$.

Proof: (The calculations to follow are of course standard but are given for lack of a reference.) If $\{\mathrm{B}\mathbf{Z}/47\mathbf{Z}\} = 1$, then by the theorem $\lambda_t(\{\mu_{47}\}) = 1$ in $K_0(\mathrm{Ab}_{(\mathbf{Z}/47\mathbf{Z})^*})$ and in particular $\{\mu_{47}\} = 0$. Now, the maximal order of $\mathbf{Z}[\mathbf{Z}/23]$ is $\mathbf{Z} \times \mathbf{Z}[\zeta_{23}]$, where ζ_{23} is a primitive 23'rd root of unity. Furthermore, the map $K_0(\mathbf{Z} \times \mathbf{Z}[\zeta_{23}]) \to K_0(\mathbf{Z}[\mathbf{Z}/23])$ induced by the inclusion is an isomorphism (cf., [C-R87, Cor. 39:26]). The action of $\mathbf{Z}[\mathbf{Z}/23]$ factors through the inclusion $\mathbf{Z}[\mathbf{Z}/23] \subseteq \mathbf{Z} \times \mathbf{Z}[\zeta_{23}]$ (by direct inspection or as the index of that inclusion is relatively prime to 47) and by the isomorphism above it is enough to show that $\{\mu_{47}\} \neq 0 \in K_0(\mathbf{Z} \times \mathbf{Z}[\zeta_{23}]) = K_0(\mathbf{Z}[\zeta_{23}])$. Now, picking a generator ζ for μ_{47} we get a surjective $\mathbf{Z}/23$-equivariant map $\mathbf{Z}[\zeta_{23}] \to \mu_{47}$ mapping ζ_{23}^i to ζ^{ia}, where $a \in (\mathbf{Z}/47)^*$ is an element of order 23, and its kernel is an ideal I of $\mathbf{Z}[\zeta_{23}]$ so that $\{\mu_{47}\} = \{\mathbf{Z}[\zeta_{23}]\} - \{I\}$ and as $K_0(\mathbf{Z}[\zeta_{23}]) = \mathbf{Z} \bigoplus \mathrm{Pic}(\mathbf{Z}[\zeta_{23}])$ so that $\{\mu_{47}\} \neq 0$ precisely when I is non-principal. That it is non-principal is [Sw69, Thm. 3]. □

Remark: i) The results on the classes in $K_0(\mathrm{Ab}_G)$ are somewhat disappointing from the point of view of finding invariants as they say in particular that the values of the NS^k are determined by the *class* of the character group of the torus resp. finite group scheme. They must of course be determined by the character group itself but the higher invariants could possibly contain more information than the first one (which is essentially the class of the character group).

In the case of the invariants in $L_0(\mathrm{Ab}_{\mathbf{k}})$ the situation seems to be different however. To begin with for an n-dimensional torus U with cocharacter group M it is not necessarily the case that $\mathrm{NS}^{n-1}(\{U\}) = -\{M\}$ in $L_0(\mathrm{Ab}_{\mathbf{k}})$. Indeed, consider a field for which we have a quotient $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k}) \to \Sigma_n$ with kernel K and let $M := \mathbf{Z}[\{1, 2, \ldots, n\}]/\mathbf{Z}([1] + \cdots + [n])$. Here we can compactify the corresponding torus by using the standard Σ_n-action on $\mathbf{P}^{n-1}$ and twisting by

the Σ_n-torsor over $\mathbf{k}$ given by the surjection above. In other words, the generators of the rays of the fan are exactly the residues of the $[i]$ (and the cones are generated by the images of $\{1,2,\dots,n\}$ minus one element). Then Theorem 5.6 gives that $\mathrm{NS}^{n-2}(U)$ is equal to $\{\mathbf{Z}\}-\{\mathbf{Z}[\{1,2,\dots,n\}]\}$. This is not equal $-\{M\}$ in $L_0(\mathrm{Ab}_{\mathbf{k}})$. Indeed, if that were the case we could apply $\mathrm{Hom}_{\mathbf{Z}}(-,\mathbf{Z})\colon L_0(\mathrm{Ab}_{\mathbf{k}}) \to L_0(\mathrm{Ab}_{\mathbf{k}})$ to the equality and get that $\{N\}=\{\mathbf{Z}[\{1,2,\dots,n\}]\}-\{\mathbf{Z}\}$. Then applying $(-)_K$ shows that we would have the same equality in $L_0(\mathrm{Ab}_{\Sigma_n})$. However, we have a map $H^1(\Sigma_n,-)\colon L_0(\mathrm{Ab}_{\Sigma_n}) \to L_0(\mathrm{Ab})$ and the right hand side maps to zero whereas $H^1(\Sigma_n,N)\neq 0$. Indeed, we have an exact sequence $0\to\mathbf{Z}\longrightarrow\mathbf{Z}[\{1,2,\dots,n\}]\longrightarrow N\to 0$ and the long exact sequence of cohomology shows that $H^1(\Sigma_n,N)=\mathbf{Z}/n$. In this case however we have that $\mathrm{NS}^k(\{U\})=(-1)^{n-1-k}\lambda^{n-1-k}(\mathrm{NS}^{n-2}(\{U\}))$ in $L_0(\mathrm{Ab}_{\mathbf{k}})$. This follows from a formula (see [Rö07, Rmk. 4.4]) for $\{U\}$ in $K_0(\mathrm{Spc}_{\mathbf{k}})$. I doubt that such a formula would be true in general.

ii) It seems almost miraculous that the original terms simplify to the actual results. It would be nice to have a more *a priori* proof of the final result. It is clear that one would get the same result if instead of working over a non-algebraically closed field, one worked G-equivariantly over the complex numbers so that we can use integral cohomology. We then have a G-equivariant spectral sequence

$$E_1^{i,j}=\prod_{|T|=i} H^j(X_T,\mathbf{Z}) \implies H_c^*(U)$$

coming from covering the complement of U by the X_s (and the product is over all subsets of S including the empty one). (Note that there is an implicit signature character present in the G-action on the E_1 as can be seen already in Meyer-Vietoris sequences.) This spectral sequence degenerates rationally at the E_2-level for weight reasons and as each $H_c^{2n-k}(U)$ is pure of weight $2n-2k$ (for $k\geq n$) only one row contributes to a given $H_c^{2n-k}(U)$. Provided that the E_2-term is torsion free this remains true over the integers and would immediately give the theorem. Torsion freeness seems plausible; the zeroth row for instance is just the reduced chain complex of the fan Δ which is a triangulation of the sphere.

6 Relations with invariant theory

We have seen that the problem on whether $\{\mathrm{B}G\}=1$ for a finite group G is closely related to whether $\{V/G\}=\mathbb{L}^n$ in $K_0(\mathrm{Spc}_{\mathbf{k}})$ for a faithful n-dimensional G-representation V. Considering some of the examples where we have shown that this formula is true it seems that this should be related to the rationality of the variety V/G. To see if there is a more precise relation we can look at the graded ring $\mathrm{gr}^* K_0(\mathrm{Spc}_{\mathbf{k}})$ associated to the dimension filtration of $K_0(\mathrm{Spc}_{\mathbf{k}})$. If we let $\mathrm{Bir}^n_{\mathbf{k}}$ be the set of birational equivalence classes of n-dimensional $\mathbf{k}$-varieties, then we have a surjective map $\mathbf{Z}[\mathrm{Bir}^n_{\mathbf{k}}]\to\mathrm{gr}^n K_0(\mathrm{Spc}_{\mathbf{k}})$ and it is a natural question if it is always injective. Indeed, the injectivity for all n of these maps is easily seen to be equivalent to the “basic question” (cf., [LL03, Q. 1.2]) of Larsen and Lunts. If this is so, then the relation $\{V/G\}=\mathbb{L}^n$ in $K_0(\mathrm{Spc}_{\mathbf{k}})$ does indeed imply that V/G is rational. Similarly, the graded ring associated to $K_0(\mathrm{Spc}_{\mathbf{k}})'$ is equal to the graded ring associated to $K_0(\mathrm{Spc}_{\mathbf{k}})[\mathbb{L}^{-1}]$ and is isomorphic to $(\mathrm{gr}^0 K_0(\mathrm{Spc}_{\mathbf{k}})')[\mathbb{L}^{-1}]$. If $\mathrm{SBir}_{\mathbf{k}}$ is equal to the stable equivalence classes of varieties (recall that two varieties X and Y are *stably birational* if $X\times\mathbf{A}^m$ is birational to $Y\times\mathbf{A}^n$ for suitable m and n) then we have a surjection $\mathbf{Z}[\mathrm{SBir}_{\mathbf{k}}]\to\mathrm{gr}^0 K_0(\mathrm{Spc}_{\mathbf{k}})'$ given by $[X]\mapsto\{X\}\mathbb{L}^{-\dim X}$. If it is a bijection then we can conclude that if $\{V/G\}=\mathbb{L}^n$ implies that V/G is stably birational to $\mathbf{A}^n$.

We do not know anything about the injectivity of these maps. However, we can make one observation. For a given n we may define $K_0(\mathrm{Spc}_{\mathbf{k}})^{\leq n}$ to be the group generated by algebraic spaces of dimension $\leq n$ and with the same relations as for $K_0(\mathrm{Spc}_{\mathbf{k}})$ but only involving spaces of dimension $\leq n$. Then the map $\mathbf{Z}[\mathrm{Bir}^n_{\mathbf{k}}]\to\mathrm{gr}^n K_0(\mathrm{Spc}_{\mathbf{k}})^{\leq n}$ is indeed an isomorphism. It can be checked that our calculations for $\{V/G\}$ for various V and G can actually be performed in $K_0(\mathrm{Spc}_{\mathbf{k}})^{\leq n}$, where $n=\dim V$, so that they do indeed also show that V/G is rational.

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